Efficient Recursive Digital Filters with Variable Magnitude Characteristics

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ABSTRACT

This paper considers designing in the minimax sense complementary low-pass/high-pass recursive digital filters with variable magnitude characteristics. A filter structure based on the parallel connection of two variable fractional delay all-pass filters is proposed for implementing these filters. The filter optimization is performed in two basic steps. First, an initial filter is generated using a simple design scheme. Second, this filter is used as a start-up solution for further optimization being carried out by an efficient constrained nonlinear optimization algorithm. Examples are included for illustrating the efficiency of the proposed design scheme. In addition, the performance and the complexity of the proposed variable recursive digital filters are compared with those of the other variable recursive digital filters proposed in the literature. This comparison shows that the number of multipliers for the proposed filters is less than 15 percent compared with other existing structures.

1. INTRODUCTION

IN VARIOUS digital signal processing applications, there is a need for filter with variable frequency characteristics. These applications include, e.g., sampling rate conversion, echo cancellation, phased-array antenna systems, time delay estimation, timing adjustment in all-digital receivers, modeling of music instruments, and speech coding and synthesis [1–4]. Recently, research on the optimal design and the efficient implementation of the recursive variable fractional delay filters has received considerable attention [5–8]. However, the implementation of recursive filters with variable magnitude characteristics have not gained as much attention and have been considered only by a few authors [9–11]. The purpose of this contribution is to propose a new class of magnitude-selective variable digital filters and an algorithm for their optimization.

The methods for designing variable digital filters can be broadly classified into two categories, namely, frequency transformation methods [12–15] and spectral parameter approximation methods [3, 5–11, 16]. The disadvantage of the methods, belonging into the former class, is that the edge frequencies and the ripples of the various bands cannot be independently controlled. The second class of filters does not suffer from this restriction. In this technique, the coefficients of the variable filter are expressed as the polynomials of the adjustable parameter defining the desired filter characteristic.

Variable digital filters can be constructed using either finite-impulse response or infinite-impulse response filters. From the implementation point of view, one of the best structures for recursive digital filters is a parallel connection of two all-pass filter sections. These structures have some advantageous properties, such as a reasonably low coefficient sensitivity and a low roundoff noise level.

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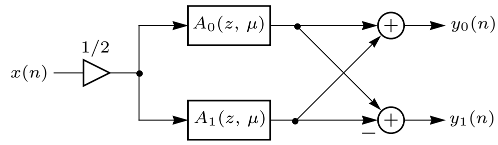

Fig. 1 Complementary low-pass/high-pass variable recursive filter pair implemented as a parallel connection of two variable fractional delay all-pass filters.

This contribution proposes a highly efficient structure for implementing complementary low-pass/high-pass variable recursive digital filter pair. This filter structure is based on the parallel connection of two variable fractional delay all-pass filters. In addition, an algorithm is proposed for optimizing the magnitude responses of these filters. Furthermore, the performance and the complexity of these filters are compared with some other variable recursive digital filters proposed in the literature showing that the number of multipliers for the proposed filters are less than 15 percent compared with other existing structures.

2. VARIABLE RECURSIVE DIGITAL FILTERS

The transfer functions of the variable digital filter pair as shown in Fig. 1 are given by

$$H_{0,1}(z,\,\mu) = \frac{1}{2}\left[A_0(z,\,\mu) \pm A_1(z,\,\mu)\right]. \tag{1}$$

Here, $H_0(z,\,\mu)$ with the plus sign and $H_1(z,\,\mu)$ with the minus sign are the low-pass and high-pass filters, respectively, and $A_0(z,\,\mu)$ and $A_1(z,\,\mu)$ are variable fractional delay all-pass filters [5, 6, 8, 17] of order N_0 and N_1, respectively, and are expressible as

$$A_k(z,\,\mu) = \frac{z^{-N_k} C_k(z^{-1},\,\mu)}{C_k(z,\,\mu)}, \tag{2a}$$

where

$$C_k(z,\,\mu) = 1 + \sum_{n=1}^{N_k} a_n^{(k)}(\mu) z^{-n} = 1 + \sum_{n=1}^{N_k}\left[\sum_{p=0}^{P} c_{pn}^{(k)}\mu^p\right] z^{-n}. \tag{2b}$$

Here, μ is an adjustable parameter in the range $[-1,\,1]$ and each coefficient in the overall filter is given as a polynomial of degree P in μ. In the case of low-pass filters, $N_0 = N_1 - 1$ or $N_0 = N_1 + 1$ so that $N_0 + N_1$, the overall order of $H_0(z)$ and $H_1(z)$ is odd. An efficient implementation for the variable fractional delay all-pass filter based on the so-called gathering structure is shown in Fig. 1 in [5].

The phase responses of the variable fractional delay all-pass filters $A_k(z,\,\mu)$ for $k=0,1$ are expressible as

$$\Theta_A^{(k)}(\omega,\,\mu) = -N_k\omega + 2\arctan\left[\frac{\sum_{n=1}^{N_k} a_n^{(k)}(\mu)\sin n\omega}{1+\sum_{n=1}^{N_k} a_n^{(k)}(\mu)\cos n\omega}\right], \tag{3}$$

whereas the magnitude responses for $H_0(z,\,\mu)$ and $H_1(z,\,\mu)$ can be written as

$$|H_0(e^{j\omega},\,\mu)| = \left|\cos\left(\tfrac{1}{2}[\Theta_A^{(0)}(\omega,\,\mu)-\Theta_A^{(1)}(\omega,\,\mu)]\right)\right| \tag{4a}$$

and

$$|H_1(e^{j\omega},\,\mu)| = \left|\sin\left(\tfrac{1}{2}[\Theta_A^{(0)}(\omega,\,\mu)-\Theta_A^{(1)}(\omega,\,\mu)]\right)\right|, \tag{4b}$$

respectively. Due to the properties of all-pass filters, the transfer functions $H_0(z,\,\mu)$ and $H_1(z,\,\mu)$ form a doubly complementary filter pair, satisfying both the all-pass complementary and the power complementary properties, that is,

$$H_0(z,\,\mu)+H_1(z,\,\mu) = z^{-(N-1)} \tag{5a}$$

$$|H_0(e^{j\omega},\,\mu)|^2+|H_1(e^{j\omega},\,\mu)|^2 = 1. \tag{5b}$$

Figure 2 shows typical phase responses for the optimized variable fractional delay all-pass filters and the corresponding magnitude responses for the complementary low-pass/high-pass recursive filter pair for $\mu=-1$ and $\mu=1$.

3. PROBLEM STATEMENT

The goal is to determine the adjustable parameters such that for each value of μ within $-1\le\mu\le1$, the magnitude response of $H_0(z,\,\mu)$ $[H_1(z,\,\mu)]$ closely approximates unity [zero] for $\omega\in[0,\,\omega_p+\alpha\mu]$ and zero [unity] for $\omega\in[\omega_s+\alpha\mu,\,\pi]$. Here, ω_p and ω_s are, respectively, the passband and stopband edge frequencies for the low-pass part of the variable digital filter for $\mu=0$, whereas α determines the desired tuning range of the filter. The doubly complementary property of Eq. (5b) guarantees that where one filter has a passband, the second one has a stopband and vice versa. In addition, for practical stopband attenuation (at least 30 dB), the passband ripple becomes very small. Therefore, the optimization of the overall complementary low-pass/high-pass filter pair can concentrate only on the stopband regions of the filters.

We state the following optimization problem:

Optimization problem: Given N_0, N_1, P, ω_p, ω_s, and α find the adjustable parameter vector Φ as given by

$$\begin{aligned}\Phi = [&c_{01}^{(0)},\ldots,c_{0N_1}^{(0)},c_{11}^{(0)},\ldots,c_{1N_1}^{(0)},\ldots,c_{P1}^{(0)},\ldots,c_{PN_1}^{(0)}\\ &c_{01}^{(1)},\ldots,c_{0N_2}^{(1)},c_{11}^{(1)},\ldots,c_{1N_2}^{(1)},\ldots,c_{P1}^{(1)},\ldots,c_{PN_2}^{(1)}]\end{aligned} \tag{6}$$

to minimize the maximum absolute value of the magnitude errors as given by

$$\epsilon = \max\{\epsilon_0(\Phi),\,\epsilon_1(\Phi)\}, \tag{7a}$$

where for $k=0,1$

$$\epsilon_k(\Phi) = \max_{-1\le\mu\le1}[\max_{\omega\in\Omega_s^{(k)}}|H_k(\Phi,\,e^{j\omega},\,\mu)|] \tag{7b}$$

subject to the constraint that the resulting filter is stable for all the values of μ within $-1\le\mu\le1$. In the above equation, the stopband regions of $H_0(z,\,\mu)$ and $H_1(z,\,\mu)$ are $\Omega_s^{(0)}=[\omega_s+\alpha\mu,\,\pi]$ and $\Omega_s^{(1)}=[0,\,\omega_p+\alpha\mu]$, respectively.

4. PROPOSED TWO-STEP DESIGN SCHEME

This section describes the proposed two-step technique for optimizing the variable recursive digital filters.

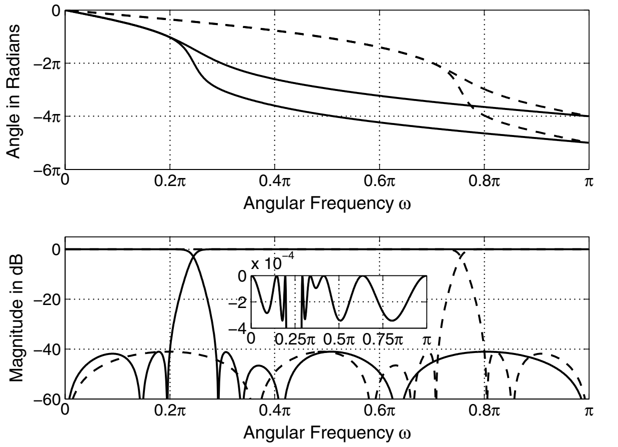

Fig. 2 The phase responses of the optimized variable fractional delay all-pass filters and the corresponding magnitude responses for the complementary low-pass/high-pass variable recursive filter pair to be considered in Example 1 for $\mu=-1$ (solid line) and $\mu=1$ (dashed-line).

4.1 Optimization Algorithm

In order to solve the optimization problem stated in the previous section we discretize the range $-1\le\mu\le1$ into the points $\mu_j\in[-1,\,1]$ for $j=1,2,\ldots,J$ and the stopband regions into the frequency points $\omega_i^{(0)}\in[\omega_s+\alpha\mu_j,\,\pi]$ and $\omega_i^{(1)}\in[0,\,\omega_p+\alpha\mu_j]$ for $i=1,2,\ldots,I$ and $j=1,2,\ldots,J$. In many cases, $I=100N$ and $J=20$ are good selections to arrive at a very accurate solution. The resulting discrete minimax problem is to find the adjustable parameter vector Φ to minimize Eq. (7a), where for $k=0,1$

$$\epsilon_k(\Phi) = \max_{\substack{1\le i\le I\\1\le j\le J}}|H_k(\Phi,\,e^{j\omega_i^{(k)}},\,\mu_j)| \tag{8}$$

subject to the condition that the roots of the $A_k(z,\,\mu_j)$'s are inside the unit circle for $\mu_j\in[-1,\,1]$ for $j=1,2,\ldots,J$.

The above problem can be solved using a constrained nonlinear optimization algorithm. For this purpose, the function `fminimax` from the optimization toolbox provided by MathWorks, Inc. [18] has been used. When using this function, the user has to provide a function that evaluates the objective function, that is, the error function to be minimized at the given frequency points as well as the gradients of the objective function with respect to the adjustable parameters. In addition, a function evaluating the constraints as well as the gradients of these constraints is required.

4.2 Algorithm for Finding an Initial Filter

The convergence of the above algorithm to the optimal solution implies a rather good initial solution for the adjustable parameters. An initial solution for further optimization can be derived by utilizing the technique proposed in [11]. In this technique, a set of J optimal fixed-coefficient filters corresponding to the above parameters μ_j for $j=1,2,\ldots,J$ are designed. Then based on the resulting coefficient sets, a polynomial approximation is derived for each of the coefficients. The design of filters with fixed coefficients can be carried out using the classical analog-filter approximations and then converting a resulting continuous-time transfer function into a corresponding discrete-time transfer function. It is well known that the odd order elliptic filter is the most selective low-pass or high-pass filter being implementable as a parallel connection of two all-pass filters (see, e.g., [19]).

4.3 Practical Considerations

The minimum value of N_0+N_1, the order of the overall filter, satisfying the specifications can be estimated using well-known

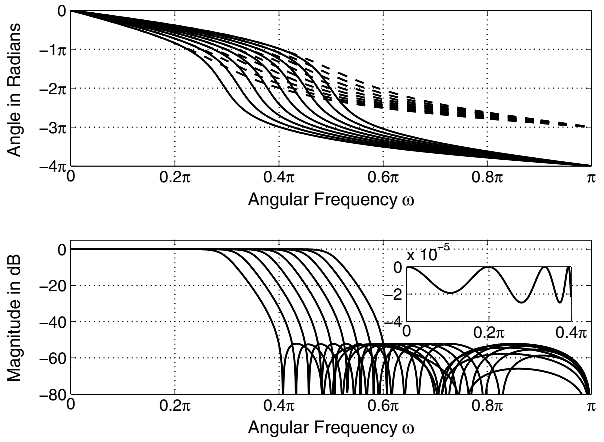

Fig. 3 The phase responses of the optimized variable fractional delay all-pass filters and the magnitude responses for the low-pass part of overall variable recursive filter in Example 2 for $\mu = -1, -0.75, -0.5, \ldots, 1$.

approximation formulas by finding the minimum odd order of an elliptic filter to meet the magnitude specifications for $\mu = 0$. If the design margin is very small, then this overall order should be increased by two. For this filter structure, there is no simple way to estimate the minimum value of P to meet the given overall criteria. However, this value can be determined by first starting from $P = 1$ and then gradually increasing this value and re-designing the filter until the specifications are met.

In order to guarantee that the resulting filter is stable, it is required that the poles of the all-pass filters are inside the unit circle for all μ_j's. The transfer functions of the $A_k(z)$'s cannot be parameterized in such a form for which the stability can be easily verified, e.g., as a cascade of first- and second-order sections. Therefore, we have used the Schür-Cohn stability test [20]. For the stability, it is required that the maximum absolute value of stability test parameters given by the Schür-Cohn stability test are smaller than unity.

The number of adders and multipliers required to implement an Nth-order fractional delay all-pass filter, as given by Eq. (2), are $(P+2)N$ and $(P+1)N$, respectively.[1] However, it has turned out that if the resulting overall filter for $\mu = 0$ is a half-band filter, that is, $\omega_p = \pi - \omega_s$, then the number of adders and multipliers required to implement an all-pass filter reduces to $\lfloor (P+1)N/2 \rfloor + N$ and $\lfloor (P+1)N/2 \rfloor$, respectively.

The implementation cost can be further reduced by constraining some of the coefficients to be equal to zero. These coefficients can be determined by fixing those values of the $c_{pn}^{(k)}$'s to be equal to zero for which $|c_{pn}^{(k)}| < \xi$ and re-optimizing the rest of the coefficient until their effects on decreasing the stopband attenuation is still tolerable. A good choice for ξ is within 0.01 and 0.1.

5. NUMERICAL EXAMPLES

This section illustrates, by means of examples, the flexibility and the effectiveness of the proposed optimization scheme. In addition, the performance and the complexity of the proposed variable recursive filters are compared with those of other existing variable recursive filters proposed in the literature.

5.1 Example 1

It is required that $\omega_p = 0.45\pi + \alpha\mu$, where $\alpha = 0.25\pi$, $\omega_s = \omega_p + 0.1\pi$, and $\delta_s \leq 0.01$ (40-dB stopband attenuation). The minimum odd order of an elliptic filter to meet the specifications for $\mu = 0$ is seven. The stopband attenuation for this

[1]The number of multipliers required to implement the multiplications by μ^p for $p = 1, 2, \ldots, P$ are not included in this figure.

Table 1 Optimized coefficient values for the variable fractional delay all-pass filters in Example 2 ($N_0 = 3$, $N_1 = 4$, and $P = 2$).

$c_{01}^{(0)} = -0.3340$	$c_{02}^{(0)} = 0.3515$	$c_{03}^{(0)} = 0$	
$c_{11}^{(0)} = 0.3312$	$c_{12}^{(0)} = -0.0062$	$c_{13}^{(0)} = 0$	
$c_{21}^{(0)} = 0$	$c_{22}^{(0)} = 0$	$c_{23}^{(0)} = 0$	
$c_{01}^{(1)} = -0.6273$	$c_{02}^{(1)} = 0.8907$	$c_{03}^{(1)} = -0.1253$	$c_{04}^{(1)} = 0.0557$
$c_{11}^{(1)} = 0.6205$	$c_{12}^{(1)} = -0.1204$	$c_{13}^{(1)} = 0.1249$	$c_{14}^{(1)} = 0$
$c_{21}^{(1)} = 0$	$c_{22}^{(1)} = 0.0582$	$c_{23}^{(1)} = 0$	$c_{24}^{(1)} = 0$

fixed-coefficient filter is -41.06 dB. In order to increase the design margin, the overall order is increased by two, that is, $N_0 = 5$ and $N_1 = 4$. For the $P = 1$ case, the stopband attenuation for the optimized filter is 19.47 dB. The given specifications are met by $P = 2$, that is, the number of multipliers required to implement all the filter coefficients is 27. For the optimized filter, the passband ripple and the stopband attenuation are $3.38 \cdot 10^{-4}$ dB and 40.58 dB, respectively, whereas the radius of the outermost pole is 0.941.

The resulting filter for $\mu = 0$ is a half-band filter and, therefore, $c_{01}^{(k)}$, $c_{21}^{(k)}$, $c_{12}^{(k)}$, $c_{03}^{(k)}$, $c_{23}^{(k)}$, and $c_{14}^{(k)}$ for $k = 1, 2$, as well as $c_{05}^{(0)}$ and $c_{25}^{(0)}$ can be fixed to be zero-valued. Further, the absolute values of $c_{24}^{(0)}$, $c_{15}^{(0)}$, and $c_{24}^{(1)}$ are smaller than 0.01. Forcing also these coefficients to be zero-valued and re-optimizing the overall filter results in a $4.04 \cdot 10^{-4}$-dB passband ripple and a 40.31-dB stopband attenuation. In this case, the number of multipliers required to implement all the filter coefficients is only ten. Some phase responses for the optimized variable fractional delay all-pass filters and the corresponding magnitude responses for the complementary low-pass/high-pass variable recursive filter pair are shown in Fig 1.

In this and in the following examples, the optimization has been performed with $I = 100N$ and $J = 20$, whereas the resulting passband ripple and the stopband attenuation as well as the other figures of merit are evaluated with $I = 2^{15}$ and $J = 50$.

5.2 Example 2

In [10], it is required that $\omega_p = 0.3\pi + \alpha\mu$, where $\alpha = 0.1\pi$, $\omega_s = \omega_p + 0.2\pi$, and $\delta_s \leq 0.01$. The passband ripple and the stopband attenuation for the variable recursive digital filter in [10] are 0.062 dB and 38.29 dB, respectively, whereas the total number of multipliers needed to implement the overall filter is 117.[2]

For the proposed structure, these specifications are met by $N_0 = 3$, $N_1 = 4$, and $P = 2$, that is, the overall number coefficients is only 21. The passband ripple and the stopband attenuation for the optimized filter are $2.81 \cdot 10^{-5}$ dB and 51.97 dB, respectively. When those $c_{pn}^{(k)}$'s for which the maximum absolute value is smaller than 0.1 are fixed to be zero-valued and the overall filter is re-optimized then the corresponding figures are $2.65 \cdot 10^{-4}$ dB and 42.22 dB, respectively. In this case, the number of multipliers is 12.

The optimized coefficients $c_{pn}^{(0)}$ for $n = 1, 2, 3$ and $p = 0, 1, 2$ and $c_{pn}^{(1)}$ for $n = 1, 2, 3, 4$ and $p = 0, 1, 2$ are shown in Table 1. The phase responses for the optimized variable fractional delay all-pass filters and the magnitude responses for the low-pass part of the overall variable recursive filter pair are depicted in Fig. 3. In addition, the passband details of the magnitude response are shown in this figure in the $\mu = 1$ case.

5.3 Example 3

In [9, 11], the desired magnitude characteristics of a variable low-pass filter are stated as follows: unity for $\omega \in [0,\ 0.26\pi + \Psi]$ and

[2]It should be pointed out that the filter structure proposed in [10] is free from transients at the filter output during the changes in the filter parameters.

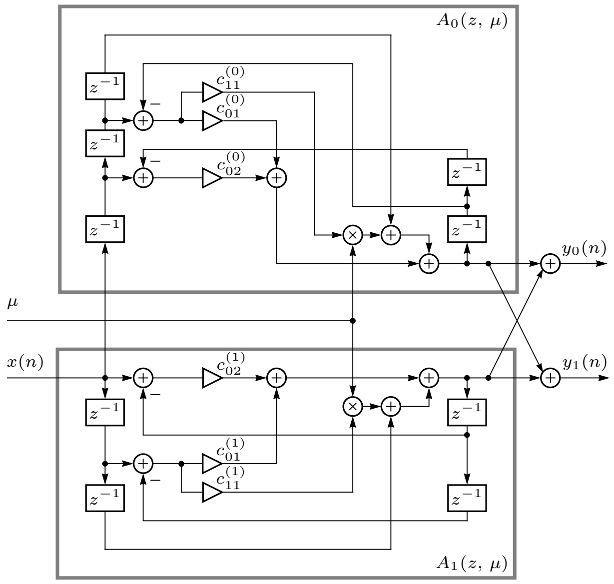

Fig. 4 Efficient implementation for the optimized complementary variable recursive filter pair in Example 3.

zero for $\omega \in [0.5\pi + \Psi,\ \pi]$. Here, the parameter Ψ is in the range $[-0.16\pi,\ 0.16\pi]$. In [9], the numerator for a recursive filter used to meet the specifications is of order four and the denominator is a cascade of two second-order sections, whereas the degree of the polynomial approximation is four ($P = 4$), that is, the overall number of coefficients is 45. For the optimized filter in [9], the passband ripple and the stopband attenuation are $7.24 \cdot 10^{-2}$ dB and 23.65 dB, respectively.

For the proposed filter structure, after fixing some coefficients equal to zero, only six coefficients are needed with $N_0 = 3$, $N_1 = 2$, and $P = 2$ to meet approximately the same specifications. The passband ripple and the stopband attenuation are $9.52 \cdot 10^{-3}$ dB and 26.60 dB, respectively. An efficient implementation for the optimized overall filter is depicted in Fig. 4. For simplicity, the multiplication of the input signal by $1/2$ is omitted from this figure.

5.4 Summary

The summary for the filters optimized in Examples 1, 2, and 3 is shown in Table 2. In this table, A_p and A_s denote the passband ripple and the stopband attenuation in decibels, respectively, whereas $r_{\max}$, N_{M}, and N_{O} denote, respectively, the radius of outermost the pole, the number of multipliers required to implement all the filter coefficients, and the number of arithmetic operations per sample required for implementing the overall filter pair.

6. CONCLUSION

This paper has proposed a new structure for implementing complementary low-pass/high-pass variable recursive digital filters. It has been shown that the tuning range of the proposed structure is very wide without considerable degradation of the filters attenuation characteristics. In addition, it has been shown that significant savings in the implementation cost are achieved by using this structure. Furthermore, a two-step algorithm has been proposed for optimizing these filters.

Table 2 Summary of filter designs in Examples 1, 2, and 3.

		A_p (dB)	A_s (dB)	$r_{\max}$	N_{M}	N_{O}
Ex. 1	Proposed	$3.38 \cdot 10^{-4}$	40.58	0.941	27	69
	Proposed	$4.04 \cdot 10^{-4}$	40.31	0.941	10	35
Ex. 2	Ex. 1 in [10]	$6.2 \cdot 10^{-2}$	38.29	–	117	244
	Proposed	$2.81 \cdot 10^{-5}$	51.97	0.895	21	55
	Proposed	$2.65 \cdot 10^{-4}$	42.22	0.872	12	34
Ex. 3	Ex. 1 in [9]	$7.24 \cdot 10^{-2}$	23.65	0.825	45	99
	Proposed	$3.46 \cdot 10^{-3}$	30.96	0.833	10	29
	Proposed	$9.52 \cdot 10^{-3}$	26.60	0.800	6	20

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