Systematic Algorithm for Designing Multiplierless Computationally Efficient Recursive Decimators and Interpolators

Juha Yli-Kaakinen and Tapio Saramäki

Institute of Signal Processing
Tampere University of Technology
Finland

4th International Symposium on
Image and Signal Processing and Analysis

Outline

1. Recursive Decimators and Interpolators
   - Introduction to Nth-band Filters
   - Commutative Structures for Nth-band Filters
   - Single-stage and Multistage Decimators and Interpolators
   - Definition of "Multiplierless" Filters
2. Filter Optimization
   - Optimization Problem
   - Optimization Algorithm
3. Examples
   - Illustrative Example

Recursive *N*th-band Filters

Why Recursive *N*th-band Filters?

Best structures for decimation and interpolation:

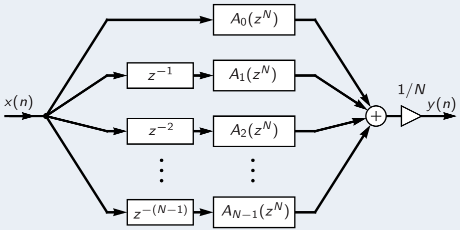
- Low multiplication rate
- Very low sensitivity when implemented using wave digital structures

Transfer function:

$$H(z) = \frac{1}{N}\sum_{n=0}^{N-1} z^{-n} A_n(z^N),$$

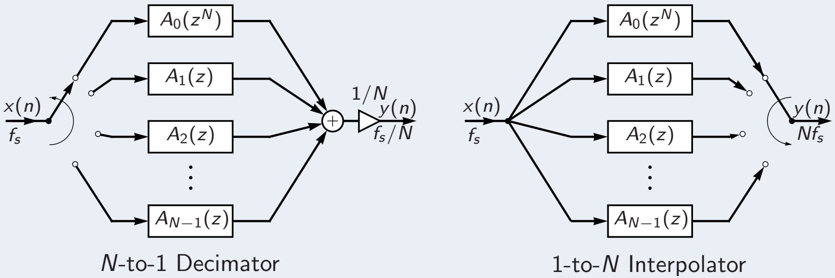
where $A_n(z)$'s are all-pass transfer functions.

The polyphase structure:



Commutative Structures for Interpolation & Decimation

Highly efficient structures



$A_n(z^N)$'s are realized at the lower sampling rate

$\Rrightarrow$ The computational workload is reduced by N.

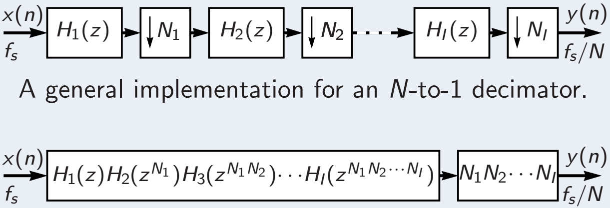
Multistage Decimators and Interpolators

If the sampling rate conversion ratio can be factored into the product

$$N = \prod_{i=1}^{I} N_i,$$

where $N_1, N_2, \ldots, N_I$ are integers, then the decimation can be implemented using I stages.

Multistage Decimator



A general implementation for an N-to-1 decimator.

Its single-stage equivalent.

Multiplierless Filters

Definition: "Multiplierless" filters

Multiplication of a data sample by each coefficient value is carried out by using the sequence of shifts and adds (or subtracts).

Example

$h(n) = 2^{-1} - 2^{-3} + 2^{-5} = 0.40625$

Desired coefficient representation form

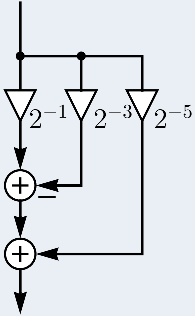
$$\sum_{r=1}^{R} a_r 2^{-P_r}$$

each a_r is 1 or -1 and P_r's are integers.

Typically R is two or three

Irregular coefficient representation form!

Implementation



Optimization Problem

Statement of the problem

Given the filter specifications

a) the passband edge ω_p,
b) the sampling ratio alternation factor N,
c) the number of stage I, and
d) the stopband edge.

Find the filter parameters

1) the orders of the subfilters K_i's and
2) the discrete coefficient values

in such a manner that

i) the filter meets the given specifications,
ii) the implementation cost, that is, the number of adders required to implement all the coefficients is minimized.

Two-Step Optimization Algorithm

Coefficient optimization is performed in two stages:

Step 1: Use a nonlinear optimization algorithm for determining a parameter space including the feasible space where the filter meets the given criteria.

Step 2: Search the filter parameters in this space in such a manner that the resulting filter meets the given criteria with the simplest coefficient representation forms.

Property

It has been experimentally proved that the optimum finite-wordlength solution can always be found.

Particularly efficient technique for all-pass filters

Only the denominator coefficients have to be quantized.

Optimization Algorithm

Step 1: Optimization of Infinite-Precision Filters

For each filter coefficient (real pole), determine the smallest and largest values, denoted by

$$r_l^{(n)(\text{min})} \quad \text{and} \quad r_l^{(n)(\text{max})} \quad \text{for } l = 1, 2, \ldots, K_n \text{ and } n = 0, 1, \ldots, N-1,$$

so that by reoptimizing the values of the remaining coefficients the magnitude response meets the given criteria.

Step 2: Optimization of Finite-Precision Filters

After finding the infinite-precision search space, check whether in this space there exist combinations of discrete pole positions satisfying the specifications.

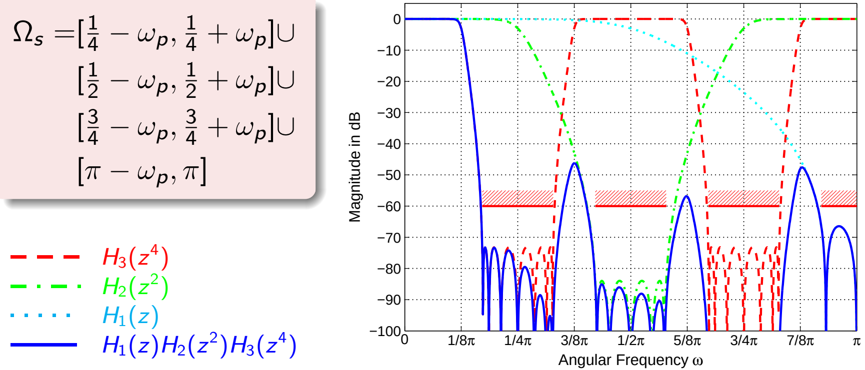
Can be done by evaluating the magnitude response for each combination of discrete coefficient values between $r_l^{(n)(\text{min})}$'s and $r_l^{(n)(\text{max})}$'s.

Illustrative Example: Infinite-Precision Optimization

Specifications: $\omega_p = 0.0785\pi = 0.628\pi/8$ and $\delta_s = 0.001$ (60 dB)

- Three-stage design: Cascade of three half-band filters
- The number of multipliers required to meet the specifications for $H_3(z^4)$, $H_2(z^2)$, and $H_1(z)$ are 3, 2, and 1, respectively.

$$\Omega_s = [\tfrac{1}{4} - \omega_p, \tfrac{1}{4} + \omega_p] \cup [\tfrac{1}{2} - \omega_p, \tfrac{1}{2} + \omega_p] \cup [\tfrac{3}{4} - \omega_p, \tfrac{3}{4} + \omega_p] \cup [\pi - \omega_p, \pi]$$



Illustrative Example: Finite-Precision Optimization

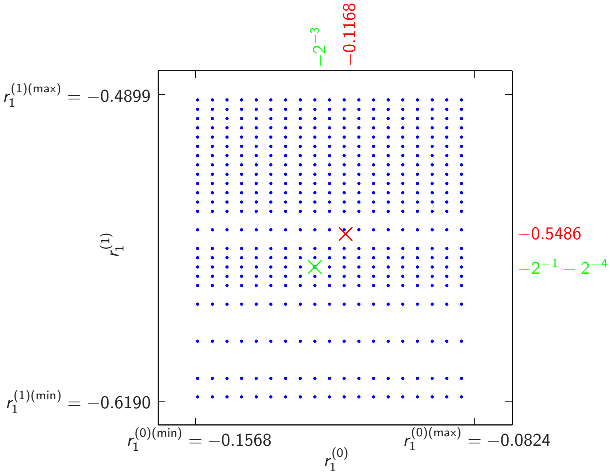
The smallest and largest values for the parameters

$H_3(z^4)$	$r_1^{(0)(\min)} = -0.1116$	$r_1^{(0)(\max)} = -0.0578$	$\Delta r_1^{(0)} = 0.0538$
	$r_2^{(0)(\min)} = -0.7711$	$r_2^{(0)(\max)} = -0.6811$	$\Delta r_2^{(0)} = 0.0900$
	$r_1^{(1)(\min)} = -0.3952$	$r_1^{(1)(\max)} = -0.2684$	$\Delta r_1^{(1)} = 0.1268$
$H_2(z^2)$	$r_1^{(0)(\min)} = -0.1568$	$r_1^{(0)(\max)} = -0.0824$	$\Delta r_1^{(0)} = 0.0744$
	$r_1^{(1)(\min)} = -0.6190$	$r_1^{(1)(\max)} = -0.4899$	$\Delta r_1^{(1)} = 0.1291$
$H_1(z)$	$r_1^{(0)(\min)} = -0.3418$	$r_1^{(0)(\max)} = -0.3366$	$\Delta r_1^{(0)} = 0.0052$

Coefficient representation: 4 power-of-two terms and 8 fractional bits

$H_3(z^4)$	14 discrete values between $r_1^{(0)(\min)}$ and $r_1^{(0)(\max)}$	9702 combinations
	21 discrete values between $r_2^{(0)(\min)}$ and $r_2^{(0)(\max)}$	
	33 discrete values between $r_1^{(1)(\min)}$ and $r_1^{(1)(\max)}$	
$H_2(z^2)$	19 discrete values between $r_1^{(0)(\min)}$ and $r_1^{(0)(\max)}$	627 combinations
	33 discrete values between $r_1^{(1)(\min)}$ and $r_1^{(1)(\max)}$	
$H_1(z)$	1 discrete value between $r_1^{(0)(\min)}$ and $r_1^{(0)(\max)}$	1 combination

Quantization of $H_2(z)$



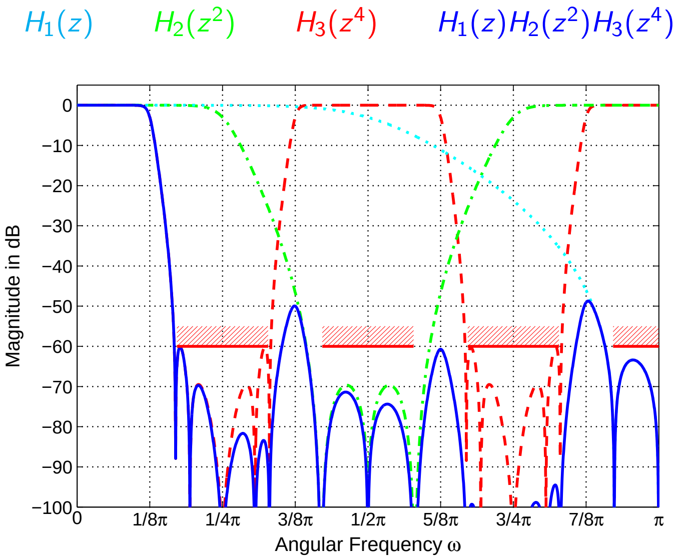
x Initial value. x optimal value.

Illustrative Example: Optimized Coefficient Values

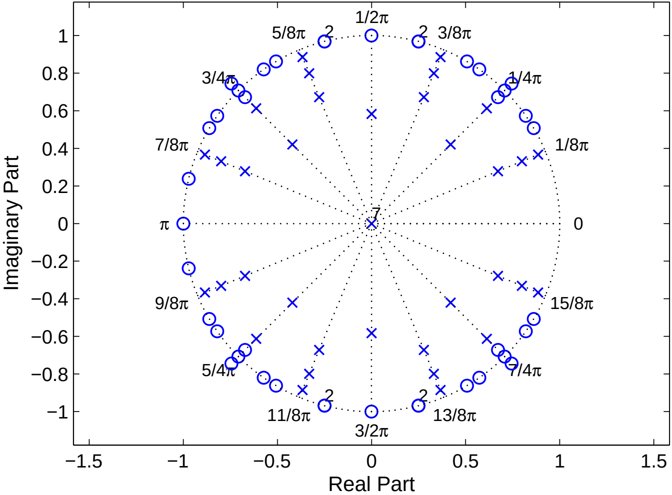
The optimal finite-precision solution between the smallest and largest values of the coefficients

$H_3(z^4)$	$r_1^{(0)(\min)} = -0.1116 <$	$-2^{-4} - 2^{-6} < r_1^{(0)(\max)} = -0.0578$
	$r_2^{(0)(\min)} = -0.7711 <$	$-1 + 2^{-2} + 2^{-5} + 2^{-7} < r_2^{(0)(\max)} = -0.6811$
	$r_1^{(1)(\min)} = -0.3952 <$	$-2^{-2} - 2^{-4} < r_1^{(1)(\max)} = -0.2684$
$H_2(z^2)$	$r_1^{(0)(\min)} = -0.1568 <$	$-2^{-3} < r_1^{(0)(\max)} = -0.0824$
	$r_1^{(1)(\min)} = -0.6190 <$	$-2^{-1} - 2^{-4} < r_1^{(1)(\max)} = -0.4899$
$H_1(z)$	$r_1^{(0)(\min)} = -0.3418 <$	$-2^{-1} + 2^{-3} + 2^{-5} + 2^{-8} < r_1^{(0)(\max)} = -0.3366$

Magnitude Response for the Optimized Three-Stage Filter



Pole-Zero Plot for the Optimized Three-Stage Filter



(b)

Comparison

Example: $\omega_p = 0.0785\pi = 0.628\pi/8$ and $\delta_s = 0.001$ (60 dB)

Structure	N_M	N_A	N_O	δ_p	δ_s
Three-stage	6	9	45	$4.91 \cdot 10^{-7}$	$0.977 \cdot 10^{-3}$
Two-stage	9	8	61	$1.07 \cdot 10^{-6}$	$0.976 \cdot 10^{-3}$
Single-stage	14	23	140	$1.70 \cdot 10^{-6}$	$0.979 \cdot 10^{-3}$

N_A give the number of adders,
N_M is the number of multipliers, and
N_O denote the overall order the filter.

(Number of multipliers for the corresponding linear-phase three-stage FIR decimator when utilizing coefficient symmetry is 33.)

Further Reading

- M. Renfors and T. Saramäki, "Recursive *N*th-band digital filters — Part I: Design and properties, Part II: Design of multistage decimators and interpolators," *IEEE Trans. Circuits Syst.*, vol. CAS-34, no. 1, pp. 24–51, Jan. 1987.
- T. Saramäki and M. Renfors, "*N*th-band filter design," in *Proc. IX European Signal Processing Conf.*, Island of Rhodes, Greece, Sept. 1998, pp. 1943–1947.
- J. Yli-Kaakinen and T. Saramäki, "Design of low-sensitivity and low-noise recursive digital filters using a cascade of low-order lattice wave digital filters," *IEEE Trans. Circuits Syst. II*, vol. 46, pp. 906–914, July 1999. [Online]. Available: `http://alpha.cc.tut.fi/~ylikaaki/CAS99.pdf`