An Algorithm for the Optimization of Adjustable Fractional-Delay All-pass Filters

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Outline

Adjustable Fractional Delay (FD) Filters

Motivation

In various DSP applications, there is a need for a delay that is a fraction of a sampling interval.

Furthermore, it is often desired that the delay value is adjustable.

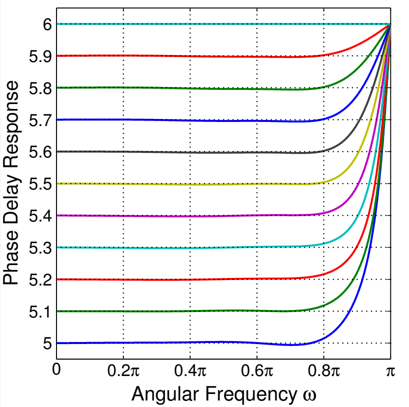
Desired frequency response of an adjustable fractional delay filter:

$$H_{\text{des}}(e^{j\omega},\ \mu) = e^{-j\omega(D+\mu)},$$

where D is a integer delay and μ is an adjustable fractional delay in the range $[-1,\ 0]$.

Example phase delay responses

$$\tau(\omega, \mu) = -\arg H(e^{j\omega}, \mu)/\omega$$



Classes of Adjustable Fractional Delay Filters

These filters can be designed either using FIR or IIR filters

Farrow structure consisting several parallel fixed FIR filters:

Outputs of these filters are multiplied with quantities depending on the value of the fractional delay.

All-pass gathering structure proposed by Makundi *et al.*:

The filter coefficient are the polynomials of the desired value of the fractional delay.

The choice for a proper class of adjustable fractional delay filters depends on the application at hand.

Adjustable FD FIR Filters vs. Adjustable FD IIR Filters

The advantages of FD IIR filters compared to their FIR counterparts

- The magnitude response is equal to unity at all the frequencies.
- The overall delay is smaller in cases requiring stringent magnitude specification.
- The complexity, that is, the number of multipliers, adders, and delays, is smaller.

Corresponding disadvantages

- The roundoff noise level at the filter output as well as the coefficient sensitivity is higher.
- Abrupt changes in their parameters may cause transients.
- Design is more complicated due to stability issues.

Gathering Structure for the Adjustable FD All-Pass Filters

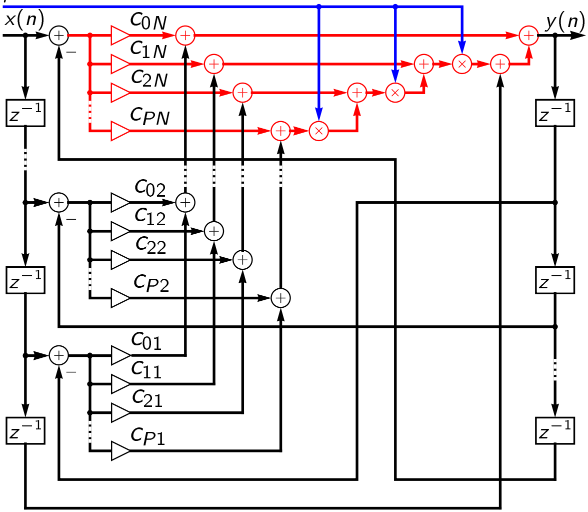
Direct-form all-pass structure

$$H(z) = \frac{z^{-N}A(z^{-1})}{A(z)}$$

Coefficients are given as a polynomial functions of degree P in μ

$$a_n(\mu) = \sum_{p=0}^{P} c_{pn}\mu^p$$

$$a_N(\mu) = c_{PN}\mu^P + \cdots + c_{2N}\mu^2 + c_{1N}\mu + c_{0N}$$



Adjustable Fractional Delay All-Pass Filters

Transfer function of the adjustable FD all-pass filter

$$H_A(z,\,\mu) = \frac{z^{-N}A(z^{-N},\,\mu)}{A(z,\,\mu)},$$

where

$$A(z,\,\mu) = 1 + \sum_{n=1}^{N} a_n(\mu) z^{-n} = 1 + \sum_{n=1}^{N} \left(\sum_{p=0}^{P} c_{pn}\mu^p \right) z^{-n}.$$

The phase delay response of the corresponding filter is expressible as

$$\tau_A(\omega,\,\mu) = -\Theta_A(\omega,\,\mu)/\omega,$$

where

$$\Theta_A(\omega,\,\mu) = -N\omega + 2\arctan\left(\frac{\sum_{n=1}^{N} a_n(\mu)\sin n\omega}{1 + \sum_{n=1}^{N} a_n(\mu)\cos n\omega} \right).$$

Optimization Problem

Optimization goal

Determine the adjustable parameters such that for each value of μ within $-1 \leq \mu \leq 0$ the phase delay closely approximates $N + \mu$.

Statement of the problem

Given the filter specifications

1. the passband edge ω_p,
2. the filter order N, and
3. the polynomial order P,

find the filter coefficient values in such a manner that

1. the maximum absolute value of the phase delay error is minimized, and
2. the resulting filter is stable for all the values of μ within $-1 \leq \mu \leq 0$.

General Two-step Optimization Technique

Filter optimization is performed in two stages:

Step 1: Design a suboptimal initial filter for further optimization using a simple design scheme.

Step 2: Optimize this initial filter using a general-purpose nonlinear optimization algorithm, giving the desired optimized solution.

If there is a systematic algorithm for finding an initial solution being close to the optimum one, then this two-step procedure finds in most cases the optimum solution very fast.

Algorithm for Finding an Initial Filter

Design technique proposed by Laakso *et al.*

In this algorithm, the weighted least-squares phase delay error as given by

$$\epsilon_{\mathrm{ls}} = \frac{1}{\pi} \int_0^{\omega_p} \frac{W(\omega)}{\omega^2} \left|\Delta\Theta(\omega)\right|^2 \mathrm{d}\omega$$

is minimized.

Here, $W(\omega)$ is a weighting function and $\Delta\Theta(\omega)$ is the difference between the desired and actual phase response.

This error is minimized for a set of μ_j's and a polynomial approximation is derived for the coefficients based on the resulting coefficient sets.

Optimization Algorithm

Discretize the passband region into frequency points

$$\omega_i \in [0,\ \omega_p] \quad \text{for } i = 1, 2, \ldots, I$$

and the range $-1 \leq \mu \leq 0$ into the points

$$\mu_j \in [-1,\ 0] \quad \text{for } j = 1, 2, \ldots, J.$$

$I = 10N$ and $J = 10$ are good choices to arrive at an accurate solution.

Find the adjustable parameter vector Φ containing the filter coefficients to minimize

$$\epsilon = \max_{\substack{1 \leq i \leq I \\ 1 \leq j \leq J}} \left| \tau_A(\Phi,\ \omega_i,\ \mu_j) - (N + \mu_j) \right|$$

subject to the condition that the roots of $A(z,\ \mu_j)$ are inside the unit circle for $\mu_j \in [-1,\ 0]$ for $j = 1, 2, \ldots, J$.

Can be solved using `fminimax` from the MATLAB's optimization toolbox.

Numerical Example

Example

Filter Specifications: $\Omega_p = [0,\ 0.75\pi]$ and $\delta_p \leq 0.01$.

Solution

The specifications are met by $N = 4$ and $P = 2$, that is, the number of multipliers is only eight.

The resulting phase delay error is 0.008 94 whereas the radius of the outermost pole is 0.994 30.

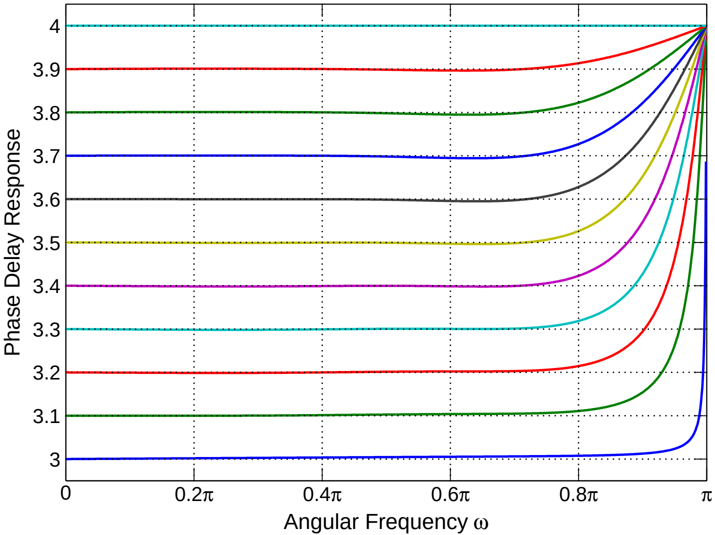
The CPU time required for the optimization is 5 s.

Optimized coefficient values

$$c_{11} = -0.92460 \quad c_{12} = 0.36268 \quad c_{13} = -0.10800 \quad c_{14} = 0.02171$$

$$c_{21} = 0.06555 \quad c_{22} = 0.35922 \quad c_{23} = -0.10741 \quad c_{24} = 0.02164$$

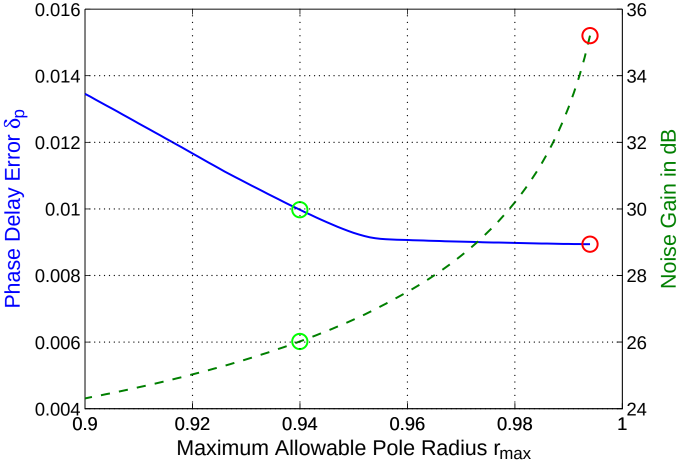
Phase Delay Responses for the Optimized Filter



Phase Delay Error & Noise Gain as a Function of Maximum Pole Radius

Design margin can be used for reducing the noise gain

$G_{\text{noise}} = 35.4$ dB without radius constraint, $G_{\text{noise}} = 26$ dB$|r_{\max} = 0.94$.



Phase Delay Errors as Functions of the Passband Edge

For estimating the filter order N and the polynomial degree P

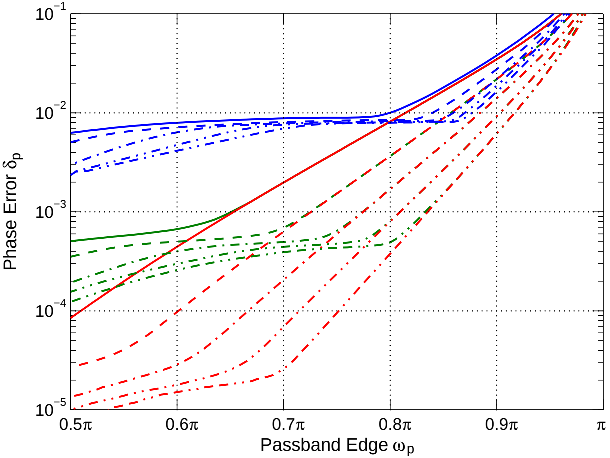
Example: $\omega_p = 0.8\pi$ and $\delta_p = 10^{-3} \Longrightarrow N = 7$ and $P = 3$.

Filter order

$N = 4$ ——
$N = 5$ – –
$N = 6$ · –
$N = 7$ · · –
$N = 8$ · · – –

Polyn. order

$P = 2$ (blue)
$P = 3$ (green)
$P = 4$ (red)



Comparison with Adjustable FD FIR filters

FD all-pass filters

δ_{phase}	δ_{ampl}	r_{max}	N	P	N_{Mult}	N_{Add}	N_{Delay}
0.0089	0	0.9943	4	2	8	12	8
0.0040	0	0.9781	4	3	12	16	12
0.0015	0	0.9794	5	3	15	20	15

FD FIR filters based on modified Farrow structure

δ_{phase}	δ_{ampl}	r_{max}	N	P	N_{Mult}	N_{Add}	N_{Delay}
0.0040	0.0235	–	8	3	19	26	24
0.0018	0.0095	–	10	3	23	34	32

Both the number multipliers and adders for the resulting all-pass filters are less than 50% compared with their FIR counterparts.

Further Reading

- M. Makundi, T. I. Laakso, and V. Välimaki, "Efficient tunable IIR and allpass filter structures," *Electron. Lett.*, vol. 37, no. 6, pp. 344–345, Mar. 2001.
- M. Makundi, V. Välimäki, and T. I. Laakso, "Closed-form design of tunable fractional-delay allpass filter structures," in *Proc. IEEE Int. Symp. Circuits Syst.*, vol. 4, Sydney, Australia, May 6–9 2001, pp. 434–437.
- T. I. Laakso, V. Välimäki, M. Karjalainen, and U. K. Laine, "Splitting the unit delay," *IEEE Signal Processing Mag.*, vol. 13, no. 1, pp. 30–60, Jan. 1996.