Optimal Design of Frequency-Response Masking Filters With Reduced Group Delays

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***Abstract*—In this paper, a new method for the design of optimal finite-impulse response frequency-response masking (FRM) filters with reduced passband group delays is proposed. To meet the prescribed magnitude response and group delay, the proposed design method takes into account both the magnitude error and the group delay error. The key step is the derivation of the group delay and its gradient with respect to the filter coefficients, based on which an explicit group delay constraint is formulated. By incorporating the group delay constraint into the overall optimization, FRM filters with better approximation to the prescribed reduced group delay can be obtained in comparison with a recent method by Lu and Hinamoto in 2003, as illustrated by two design examples.**

***Index Terms*—Finite-impulse response (FIR) filters, frequency-response masking (FRM), group delay, second-order cone programming (SOCP).**

I. Introduction

THE design of finite-impulse response (FIR) filters using the frequency-response masking (FRM) technique [1] has attracted much attention in the past two decades owing to its efficient implementation of FIR filters with sharp transition bands [2]–[4]. Most of the design methods are concerned with linear phase FRM filters since linear phase filters possess several desirable properties as compared to non-linear phase ones. However, linear phase FRM filters with sharp transition bands are known to have very large group delays in the entire frequency band [5], [6]. In many applications, such as microwave and radar communications, optical communication and medical imaging, etc. [7]–[11], reduced passband group delays are required for devices and digital filters to avoid long delay and to minimize distortion of the transmitted information. This has motivated some recent interest in the design of FRM filters with reduced passband group delays [6], [12]–[14].

Pioneering work in the design of suboptimal FRM filters with reduced passband group delays was presented in [12], [13]. An alternative approach was recently proposed in [6], [14] in which all subfilter coefficients were stacked as a design vector and optimized through a sequence of linear updates. With a reasonable initial design, the overall optimization problem was formulated and carried out in the semidefinite programming (SDP) framework [6] or in the second-order cone programming (SOCP) framework [14].

Similarly to conventional design methods for FIR filters with reduced group delays (see, e.g., [15], [16]), in the above design methods for FRM filters with reduced group delays [6], [12]–[14], the prescribed group delays were indirectly approximated. By converting the group delay into a phase response, the combination of magnitude and phase errors was minimized. Because of the nonlinear relationship between the group delay and its corresponding phase response, it is difficult and inflexible to approximate a given group delay accurately without an explicit group delay constraint, leading to a relatively large group delay error, noticeably towards the bandedges.

Recently, a new design method for FIR filters with reduced group delays was proposed in [17] and [18]. To directly control the group delay error, a group delay constraint was formed and incorporated into an optimization problem in the SDP framework. In this paper, we extend this idea to the more complicated FRM filter structure and propose a new method for the design of general FIR FRM filters with reduced and more accurate passband group delays.

In this proposed design method, the filter design problem is formulated by taking into account both the magnitude error and the group delay error. The key step is the derivation of the group delay of an FRM filter and its gradient with respect to the subfilter coefficients, based on which an explicit group delay constraint is formulated. Our design method works for a general multistage FRM filter with a nonlinear phase prototype filter and nonlinear phase masking filters. Note that for an FRM filter with reduced group delay, the prototype filter should be of nonlinear phase to effectively reduce the overall group delay which is mainly contributed by the prototype filter [6]. We hence assume throughout the paper that the prototype filter is of nonlinear phase. The main reason for considering nonlinear phase masking filters is that when an arbitrary constant group delay, especially a non-integer one, is desirable, nonlinear phase masking filters must be used. It should be pointed out that the proposed design method also works for FRM filters with linear phase masking filters. The motivation for considering multistage FRM filters is that multistage FRM filters are computationally more efficient than basic FRM filters in applications where very narrow transition bands are desired, such as in digital TV spectral emission masking [19], and acoustic feedback systems [20] etc. In applications where a basic FRM filter is sufficient, the proposed method can be simplified for the design of basic FRM filters.

The paper is organized as follows. The group delay of a two-stage FRM filter and its gradient with respect to the subfilter coefficients are derived in Section II. A new method for the design of two-stage FRM filters is then presented in Section III, in which the group delay constraint is formulated and incorporated into the overall optimization. In order to reduce the design time, the optimization problem is subsequently cast into the SOCP framework [21], [22]. The effectiveness and flexibility of the proposed design method are illustrated by two related examples

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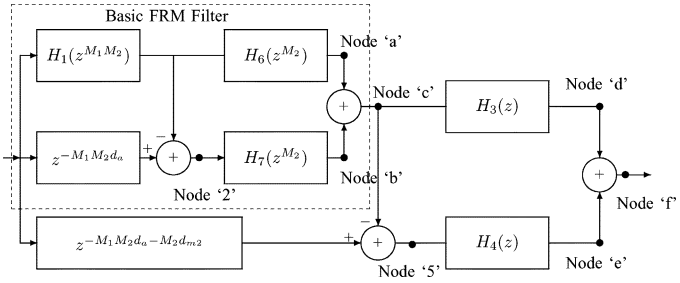

Fig. 1. Two-stage FRM filter structure.

in Section IV. Conclusions are given in Section V. Some more involved derivations are given in Appendix I for the sake of completeness, and the derivation for basic FRM filters is also given in Appendix II.

For convenience of presentation, throughout this paper, we assume that the lengths of the subfilters are all odd. With minor modifications, the method is also applicable to subfilters of even lengths, but the details are omitted here. Moreover, for the sake of notation simplicity, the frequency variable ω is omitted unless otherwise specified.

II. Group Delays and Gradients for FRM Filters

In this section, we present detailed derivation of the group delay and the gradients of the frequency response and the group delay for a two-stage FRM filter with nonlinear phase masking filters.

A. Group Delay for Two-Stage FRM Filters

As shown in Fig. 1, a two-stage FRM filter consists of a prototype filter $H_1(z)$ and two pairs of masking filters $\{H_6(z), H_7(z)\}$ and $\{H_3(z), H_4(z)\}$. The vectors of the coefficients of the subfilters are $\mathbf{h}_1$, $\mathbf{h}_6$, $\mathbf{h}_7$, $\mathbf{h}_3$, and $\mathbf{h}_4$ of lengths N_1, N_6, N_7, N_3 and N_4, respectively. Noted that M_1 and M_2 are the scaling factors of the two-stage FRM filter which affect the sharpness of the transition bands of their corresponding prototype filters [1].

Let $N_{m2} = \max\{N_6, N_7\}$ and $N_{m1} = \max\{N_3, N_4\}$, and the lengths of $\mathbf{h}_6$, $\mathbf{h}_7$, $\mathbf{h}_3$, and $\mathbf{h}_4$ are adjusted appropriately by padding zeros if necessary. Denote d_a, d_{m2} and d_{m1} the group delays of $H_1(z)$, $\{H_6(z), H_7(z)\}$ and $\{H_3(z), H_4(z)\}$, respectively. Let $A_1e^{j\phi_1}$, $A_6e^{j\phi_6}$, $A_7e^{j\phi_7}$, $A_3e^{j\phi_3}$ and $A_4e^{j\phi_4}$ be the frequency responses of $H_1(z^{M_1M_2})$, $H_6(z^{M_2})$, $H_7(z^{M_2})$, $H_3(z)$ and $H_4(z)$, respectively. We have $A_i = \sqrt{(\mathbf{h}_i^T\mathbf{c}_i)^2 + (\mathbf{h}_i^T\mathbf{s}_i)^2}$ and $\phi_i = \arctan 2(\mathbf{h}_i^T\mathbf{s}_i, \mathbf{h}_i^T\mathbf{c}_i)$ $(i = 1, 6, 7, 3, 4)$, where

$$\mathbf{c}_1 = [1 \;\; \cos M_1M_2\omega \;\; \ldots \;\; \cos(N_1-1)M_1M_2\omega]^T$$
$$\mathbf{s}_1 = [0 \;\; -\sin M_1M_2\omega \;\; \ldots \;\; -\sin(N_1-1)M_1M_2\omega]^T$$
$$\mathbf{c}_l = \left[\cos\frac{N_{m2}-N_l}{2}M_2\omega \;\; \ldots \;\; \cos\frac{N_{m2}-1}{2}M_2\omega \;\; \ldots \;\; \cos\frac{N_{m2}+N_l-2}{2}M_2\omega\right]^T, \qquad l = 6, 7$$
$$\mathbf{s}_l = \left[-\sin\frac{N_{m2}-N_l}{2}M_2\omega \;\; \ldots \;\; -\sin\frac{N_{m2}-1}{2}M_2\omega \;\; \ldots \;\; -\sin\frac{N_{m2}+N_l-2}{2}M_2\omega\right]^T, \qquad l = 6, 7$$
$$\mathbf{c}_q = \left[\cos\frac{N_{m1}-N_q}{2}\omega \;\; \ldots \;\; \cos\frac{N_{m1}-1}{2}\omega \;\; \ldots \;\; \cos\frac{N_{m1}+N_q-2}{2}\omega\right]^T, \qquad q = 3, 4$$
$$\mathbf{s}_q = \left[-\sin\frac{N_{m1}-N_q}{2}\omega \;\; \ldots \;\; -\sin\frac{N_{m1}-1}{2}\omega \;\; \ldots \;\; -\sin\frac{N_{m1}+N_q-2}{2}\omega\right]^T, \qquad q = 3, 4$$

and

$$\arctan 2(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{for } x \geq 0 \\ \arctan\left(\frac{y}{x}\right) \pm \pi & \text{otherwise.} \end{cases}$$

The frequency responses at nodes "2," "a," "b," "c," "5," "d," "e," and "f" in Fig. 1 are

$$\begin{aligned} A_2e^{j\phi_2} &= e^{-j\omega d_1} - A_1e^{j\phi_1} \\ A_ae^{j\phi_a} &= A_1A_6e^{j(\phi_1+\phi_6)} \\ A_be^{j\phi_b} &= A_2A_7e^{j(\phi_2+\phi_7)} \\ A_ce^{j\phi_c} &= A_ae^{j\phi_a} + A_be^{j\phi_b} \\ A_5e^{j\phi_5} &= e^{-j\omega d_5} - A_ce^{j\phi_c} \\ A_de^{j\phi_d} &= A_cA_3e^{j(\phi_c+\phi_3)} \\ A_ee^{j\phi_e} &= A_5A_4e^{j(\phi_5+\phi_4)} \\ A_fe^{j\phi_f} &= A_de^{j\phi_d} + A_ee^{j\phi_e} \end{aligned}$$

where $d_1 = M_1M_2d_a$ and $d_5 = d_1 + M_2d_{m2}$. Let $\mathcal{D}_2$, $\mathcal{D}_c$, $\mathcal{D}_5$, $\mathcal{D}_f$ and $\mathcal{N}_2, \mathcal{N}_c, \mathcal{N}_5, \mathcal{N}_f$ be the real and imaginary parts of $A_2e^{j\phi_2}$, $A_ce^{j\phi_c}$, $A_5e^{j\phi_5}$, and $A_fe^{j\phi_f}$, respectively, i.e.,

$$\begin{aligned} \mathcal{N}_2 &= -\sin d_1\omega - A_1\sin\phi_1 \\ \mathcal{D}_2 &= \cos d_1\omega - A_1\cos\phi_1 \\ \mathcal{N}_c &= A_a\sin\phi_a + A_b\sin\phi_b \\ \mathcal{D}_c &= A_a\cos\phi_a + A_b\cos\phi_b \\ \mathcal{N}_5 &= -\sin d_5\omega - A_c\sin\phi_c \\ \mathcal{D}_5 &= \cos d_5\omega - A_c\cos\phi_c \\ \mathcal{N}_f &= A_d\sin\phi_d + A_e\sin\phi_e \\ \mathcal{D}_f &= A_d\cos\phi_d + A_e\cos\phi_e. \end{aligned}$$

We have

$$\begin{aligned} A_2 &= \sqrt{\mathcal{D}_2^2 + \mathcal{N}_2^2}, & \phi_2 &= \arctan 2(\mathcal{N}_2, \mathcal{D}_2) \\ A_c &= \sqrt{\mathcal{N}_c^2 + \mathcal{D}_c^2}, & \phi_c &= \arctan 2(\mathcal{N}_c, \mathcal{D}_c) \\ A_5 &= \sqrt{\mathcal{D}_5^2 + \mathcal{N}_5^2}, & \phi_5 &= \arctan 2(\mathcal{N}_5, \mathcal{D}_5) \\ A_f &= \sqrt{\mathcal{N}_f^2 + \mathcal{D}_f^2}, & \phi_f &= \arctan 2(\mathcal{N}_f, \mathcal{D}_f). \end{aligned}$$

Given the phase response ϕ_f of the two-stage FRM filter, the group delay is defined as $\tau_f(\omega) = -(d\phi_f/d\omega)$. Using the derivative identity $(d/d\omega)\arctan 2(y, x) = (d/d\omega)\tan^{-1}(y/x) = (1/(1 + (y/x)^2))(d(y/x)/d\omega)$, we have

$$\tau_f(\omega) = -\frac{1}{\mathcal{D}_f^2 + \mathcal{N}_f^2}\left(\frac{d\mathcal{N}_f}{d\omega}\mathcal{D}_f - \mathcal{N}_f\frac{d\mathcal{D}_f}{d\omega}\right) \qquad (1)$$

where $d\mathcal{N}_f/d\omega$ and $d\mathcal{D}_f/d\omega$ are functions of $dA_d/d\omega$, $d\phi_d/d\omega$, $dA_e/d\omega$ and $d\phi_e/d\omega$ given by

$$\frac{d\mathcal{N}_f}{d\omega} = \frac{dA_d}{d\omega}\sin\phi_d + A_d\cos\phi_d\frac{d\phi_d}{d\omega} + \frac{dA_e}{d\omega}\sin\phi_e + A_e\cos\phi_e\frac{d\phi_e}{d\omega} \tag{2a}$$

$$\frac{d\mathcal{D}_f}{d\omega} = \frac{dA_d}{d\omega}\cos\phi_d - A_d\sin\phi_d\frac{d\phi_d}{d\omega} + \frac{dA_e}{d\omega}\cos\phi_e - A_e\sin\phi_e\frac{d\phi_e}{d\omega}. \tag{2b}$$

Subsequently, $dA_d/d\omega$, $d\phi_d/d\omega$, $dA_e/d\omega$ and $d\phi_e/d\omega$ as functions of $dA_i/d\omega$ and $d\phi_i/d\omega$ ($i = 1, 6, 7, 3, 4$) can be obtained as

$$\begin{aligned}
\frac{dA_d}{d\omega} &= \frac{dA_c}{d\omega}A_3 + A_c\frac{dA_3}{d\omega}\\
\frac{d\phi_d}{d\omega} &= \frac{d\phi_c}{d\omega} + \frac{d\phi_3}{d\omega}\\
\frac{dA_e}{d\omega} &= \frac{dA_5}{d\omega}A_4 + A_5\frac{dA_4}{d\omega}\\
\frac{d\phi_e}{d\omega} &= \frac{d\phi_5}{d\omega} + \frac{d\phi_4}{d\omega}\\
\frac{dA_c}{d\omega} &= \frac{1}{A_c}\left(\mathcal{N}_c\frac{d\mathcal{N}_c}{d\omega} + \mathcal{D}_c\frac{d\mathcal{D}_c}{d\omega}\right)\\
\frac{d\phi_c}{d\omega} &= \frac{1}{A_c^2}\left(\frac{d\mathcal{N}_c}{d\omega}\mathcal{D}_c - \mathcal{N}_c\frac{d\mathcal{D}_c}{d\omega}\right)\\
\frac{dA_5}{d\omega} &= \frac{1}{A_5}\left(\mathcal{N}_5\frac{d\mathcal{N}_5}{d\omega} + \mathcal{D}_5\frac{d\mathcal{D}_5}{d\omega}\right)\\
\frac{d\phi_5}{d\omega} &= \frac{1}{A_5^2}\left(\frac{d\mathcal{N}_5}{d\omega}\mathcal{D}_5 - \mathcal{N}_5\frac{d\mathcal{D}_5}{d\omega}\right)\\
\frac{d\mathcal{N}_c}{d\omega} &= \frac{dA_a}{d\omega}\sin\phi_a + A_a\cos\phi_a\frac{d\phi_a}{d\omega} + \frac{dA_b}{d\omega}\sin\phi_b + A_b\cos\phi_b\frac{d\phi_b}{d\omega}\\
\frac{d\mathcal{D}_c}{d\omega} &= \frac{dA_a}{d\omega}\cos\phi_a - A_a\sin\phi_a\frac{d\phi_a}{d\omega} + \frac{dA_b}{d\omega}\cos\phi_b - A_b\sin\phi_b\frac{d\phi_b}{d\omega}\\
\frac{d\mathcal{N}_5}{d\omega} &= -d_5\cos d_5\omega - \frac{dA_c}{d\omega}\sin\phi_c - A_c\cos\phi_c\frac{d\phi_c}{d\omega}\\
\frac{d\mathcal{D}_5}{d\omega} &= -d_5\sin d_5\omega - \frac{dA_c}{d\omega}\cos\phi_c + A_c\sin\phi_c\frac{d\phi_c}{d\omega}\\
\frac{dA_a}{d\omega} &= \frac{dA_1}{d\omega}A_6 + A_1\frac{dA_6}{d\omega}\\
\frac{d\phi_a}{d\omega} &= \frac{d\phi_1}{d\omega} + \frac{d\phi_6}{d\omega}\\
\frac{dA_b}{d\omega} &= \frac{dA_2}{d\omega}A_7 + A_2\frac{dA_7}{d\omega}\\
\frac{d\phi_b}{d\omega} &= \frac{d\phi_2}{d\omega} + \frac{d\phi_7}{d\omega}\\
\frac{dA_2}{d\omega} &= \frac{1}{A_2}\left(\mathcal{N}_2\frac{d\mathcal{N}_2}{d\omega} + \mathcal{D}_2\frac{d\mathcal{D}_2}{d\omega}\right)\\
\frac{d\phi_2}{d\omega} &= \frac{1}{A_2^2}\left(\frac{d\mathcal{N}_2}{d\omega}\mathcal{D}_2 - \mathcal{N}_2\frac{d\mathcal{D}_2}{d\omega}\right)\\
\frac{d\mathcal{N}_2}{d\omega} &= -d_1\cos d_1\omega - \frac{dA_1}{d\omega}\sin\phi_1 - A_1\cos\phi_1\frac{d\phi_1}{d\omega}\\
\frac{d\mathcal{D}_2}{d\omega} &= -d_1\sin d_1\omega - \frac{dA_1}{d\omega}\cos\phi_1 + A_1\sin\phi_1\frac{d\phi_1}{d\omega}.
\end{aligned}$$

Let $\tilde{\mathbf{s}}_i = d\mathbf{s}_i/d\omega$ and $\tilde{\mathbf{c}}_i = d\mathbf{c}_i/d\omega$. Then $dA_i/d\omega$ and $d\phi_i/d\omega$ ($i = 1, 6, 7, 3, 4$) are given by

$$\begin{aligned}
\frac{dA_i}{d\omega} &= \frac{1}{A_i}\mathbf{h}_i^T\left(\mathbf{c}_i\tilde{\mathbf{c}}_i^T + \mathbf{s}_i\tilde{\mathbf{s}}_i^T\right)\mathbf{h}_i\\
&= \frac{1}{A_i}\mathbf{h}_i^T\tilde{\mathbf{P}}_{m\omega i}\mathbf{h}_i, \qquad i = 1,6,7,3,4 \qquad (3a)\\
\frac{d\phi_i}{d\omega} &= \frac{1}{A_i^2}\mathbf{h}_i^T\left(\tilde{\mathbf{s}}_i\mathbf{c}_i^T - \tilde{\mathbf{c}}_i\mathbf{s}_i^T\right)\mathbf{h}_i\\
&= \frac{1}{A_i^2}\mathbf{h}_i^T\tilde{\mathbf{P}}_{p\omega i}\mathbf{h}_i, \qquad i = 1,6,7,3,4. \qquad (3b)
\end{aligned}$$

Note that both $\tilde{\mathbf{P}}_{m\omega i}$ and $\tilde{\mathbf{P}}_{p\omega i}$ (i = 1,6,7,3,4) are square matrices of compatible dimensions with $\mathbf{h}_i$. For notation simplicity, (3) can be re-expressed as

$$\begin{aligned}
\frac{dA_i}{d\omega} &= \frac{1}{A_i}\mathbf{h}_i^T\mathbf{P}_{m\omega i}\mathbf{h}_i\\
\frac{d\phi_i}{d\omega} &= \frac{1}{A_i^2}\mathbf{h}_i^T\mathbf{P}_{p\omega i}\mathbf{h}_i, \qquad i = 1,6,7,3,4 \qquad (4)
\end{aligned}$$

where $\mathbf{P}_{m\omega i}$ and $\mathbf{P}_{p\omega i}$ are symmetric matrices obtained by

$$\begin{aligned}
\mathbf{P}_{m\omega i} &= \frac{1}{2}\left(\tilde{\mathbf{P}}_{m\omega i} + \tilde{\mathbf{P}}_{m\omega i}^T\right)\\
\mathbf{P}_{p\omega i} &= \frac{1}{2}\left(\tilde{\mathbf{P}}_{p\omega i} + \tilde{\mathbf{P}}_{p\omega i}^T\right), \qquad i = 1,6,7,3,4. \qquad (5)
\end{aligned}$$

B. Gradients for Two-Stage FRM Filters

For an FRM filter with nonlinear phase masking filters, both the frequency response and the group delay are nonlinear with respect to the subfilter coefficients. Gradients of the frequency response and group delay are derived here, as they are the key components of the proposed design method to be presented in the next section.

For a two-stage FRM filter, the frequency response is

$$H_f(\omega) = H_c(M_2\omega)H_3(\omega) + \left(e^{-jd_5\omega} - H_c(M_2\omega)\right)H_4(\omega)$$

where

$$H_c(\omega) = H_1(M_1\omega)H_6(\omega) + \left(e^{-jM_1d_a\omega} - H_1(M_1\omega)\right)H_7(\omega).$$

The gradient of the frequency response with respect to the design vector $\mathbf{x} = [\mathbf{h}_1^T\ \mathbf{h}_6^T\ \mathbf{h}_7^T\ \mathbf{h}_3^T\ \mathbf{h}_4^T]^T$ is a complex-valued vector of dimension $N_T = N_1 + N_6 + N_7 + N_3 + N_4$ given by

$$\begin{aligned}
\mathbf{g}_m(\omega) &= \begin{bmatrix} \frac{dH_f(\omega)}{d\mathbf{h}_1} \\ \frac{dH_f(\omega)}{d\mathbf{h}_6} \\ \frac{dH_f(\omega)}{d\mathbf{h}_7} \\ \frac{dH_f(\omega)}{d\mathbf{h}_3} \\ \frac{dH_f(\omega)}{d\mathbf{h}_4} \end{bmatrix}\\
&= \begin{bmatrix} (H_3(\omega) - H_4(\omega))\,(H_6(M_2\omega) - H_7(M_2\omega))\,\mathbf{e}_1 \\ (H_3(\omega) - H_4(\omega))\,H_1(M_1M_2\omega)\mathbf{e}_6 \\ (H_3(\omega) - H_4(\omega))\left(e^{-jM_1M_2d_a\omega} - H_1(M_1M_2\omega)\right)\mathbf{e}_7 \\ H_c(M_2\omega)\mathbf{e}_3 \\ \left(e^{-jd_5\omega} - H_c(M_2\omega)\right)\mathbf{e}_4 \end{bmatrix}
\end{aligned} \tag{6}$$

where

$$\mathbf{e}_1=\left[1\ \ e^{-jM_1M_2\omega}\ \ e^{-j2M_1M_2\omega}\ \ \dots\ \ e^{-j(N_1-1)M_1M_2\omega}\right]^T$$

$$\mathbf{e}_l=\left[e^{-j\frac{N_{m2}-N_l}{2}M_2\omega}\ \ e^{-j\frac{N_{m2}-N_l+2}{2}M_2\omega}\ \ \dots\ \ e^{-j\frac{N_{m2}+N_l-2}{2}M_2\omega}\right]^T,\qquad l=6,7$$

$$\mathbf{e}_q=\left[e^{-j\frac{N_{m1}-N_q}{2}\omega}\ \ e^{-j\frac{N_{m1}-N_q+2}{2}\omega}\ \ \dots\ \ e^{-j\frac{N_{m1}+N_q-2}{2}\omega}\right]^T,\qquad q=3,4.$$

Using the group delay in (1), the gradient of the group delay with respect to $\mathbf{x}$ is a real-valued vector of dimension N_T given by

$$\mathbf{g}_\tau(\omega)=\begin{bmatrix}\frac{d\tau_f(\omega)}{d\mathbf{h}_1}\\ \frac{d\tau_f(\omega)}{d\mathbf{h}_6}\\ \frac{d\tau_f(\omega)}{d\mathbf{h}_7}\\ \frac{d\tau_f(\omega)}{d\mathbf{h}_3}\\ \frac{d\tau_f(\omega)}{d\mathbf{h}_4}\end{bmatrix} \tag{7}$$

where $d\tau_f(\omega)/d\mathbf{h}_i$ (i = 1,6,7,3,4), can be expressed as

$$\begin{aligned}\frac{d\tau_f(\omega)}{d\mathbf{h}_i}=&\frac{2}{\left(\mathcal{D}_f^2+\mathcal{N}_f^2\right)^2}\left(\mathcal{D}_f\frac{d\mathcal{D}_f}{d\mathbf{h}_i}+\mathcal{N}_f\frac{d\mathcal{N}_f}{d\mathbf{h}_i}\right)\\&\times\left(\frac{d\mathcal{N}_f}{d\omega}\mathcal{D}_f-\mathcal{N}_f\frac{d\mathcal{D}_f}{d\omega}\right)-\frac{1}{\mathcal{D}_f^2+\mathcal{N}_f^2}\\&\times\left(\frac{d^2\mathcal{N}_f}{d\omega d\mathbf{h}_i}\mathcal{D}_f+\frac{d\mathcal{N}_f}{d\omega}\frac{d\mathcal{D}_f}{d\mathbf{h}_i}\right.\\&\left.-\frac{d\mathcal{N}_f}{d\mathbf{h}_i}\frac{d\mathcal{D}_f}{d\omega}-\mathcal{N}_f\frac{d^2\mathcal{D}_f}{d\omega d\mathbf{h}_i}\right),\ i=1,6,7,3,4.\end{aligned} \tag{8}$$

Components of $d\mathcal{N}_f/d\mathbf{h}_i$, $d\mathcal{D}_f/d\mathbf{h}_i$ $d^2\mathcal{N}_f/d\omega d\mathbf{h}_i$ and $d^2\mathcal{D}_f/d\omega d\mathbf{h}_i$ (i = 3, 4) are given by

$$\frac{d\mathcal{N}_f}{d\mathbf{h}_i}=\frac{dA_q}{d\mathbf{h}_i}\sin\phi_q+A_q\cos\phi_q\frac{d\phi_q}{d\mathbf{h}_i} \tag{9a}$$

$$\frac{d\mathcal{D}_f}{d\mathbf{h}_i}=\frac{dA_q}{d\mathbf{h}_i}\cos\phi_q-A_q\sin\phi_q\frac{d\phi_q}{d\mathbf{h}_i} \tag{9b}$$

$$\frac{dA_q}{d\mathbf{h}_i}=A_l\frac{dA_i}{d\mathbf{h}_i},\qquad \frac{d\phi_q}{d\mathbf{h}_i}=\frac{d\phi_i}{d\mathbf{h}_i} \tag{9c}$$

$$\begin{aligned}\frac{d^2\mathcal{N}_f}{d\omega d\mathbf{h}_i}=&\frac{d^2A_q}{d\omega d\mathbf{h}_i}\sin\phi_q+\frac{dA_q}{d\omega}\cos\phi_q\frac{d\phi_q}{d\mathbf{h}_i}\\&+\frac{dA_q}{d\mathbf{h}_i}\cos\phi_q\frac{d\phi_q}{d\omega}-A_q\sin\phi_q\frac{d\phi_q}{d\mathbf{h}_i}\frac{d\phi_q}{d\omega}\\&+A_q\cos\phi_q\frac{d^2\phi_q}{d\omega d\mathbf{h}_i}\end{aligned} \tag{9d}$$

$$\begin{aligned}\frac{d^2\mathcal{D}_f}{d\omega d\mathbf{h}_i}=&\frac{d^2A_q}{d\omega d\mathbf{h}_i}\cos\phi_q-\frac{dA_q}{d\omega}\sin\phi_q\frac{d\phi_q}{d\mathbf{h}_i}\\&-\frac{dA_q}{d\mathbf{h}_i}\sin\phi_q\frac{d\phi_q}{d\omega}-A_q\cos\phi_q\frac{d\phi_q}{d\mathbf{h}_i}\frac{d\phi_q}{d\omega}\\&-A_q\sin\phi_q\frac{d^2\phi_q}{d\omega d\mathbf{h}_i}\end{aligned} \tag{9e}$$

$$\frac{d^2A_q}{d\omega d\mathbf{h}_i}=\frac{dA_l}{d\omega}\frac{dA_i}{d\mathbf{h}_i}+A_l\frac{d^2A_i}{d\omega d\mathbf{h}_i},\qquad \frac{d^2\phi_q}{d\omega d\mathbf{h}_i}=\frac{d^2\phi_i}{d\omega d\mathbf{h}_i} \tag{9f}$$

for $(i,l,q)=(3,c,d)$ or $(i,l,q)=(4,5,e)$.

Similarly, $d\mathcal{N}_f/d\mathbf{h}_i$, $d\mathcal{D}_f/d\mathbf{h}_i$, $d^2\mathcal{N}_f/d\omega d\mathbf{h}_i$ and $d^2\mathcal{D}_f/d\omega d\mathbf{h}_i$ (i = 1, 6, 7) can be derived as functions of $dA_i/d\omega$, $d\phi_i/d\omega$, $dA_i/d\mathbf{h}_i$, $d\phi_i/d\mathbf{h}_i$, $d^2A_i/d\omega d\mathbf{h}_i$ and $d^2\phi_i/d\omega d\mathbf{h}_i$ (i = 1, 6, 7, 3, 4), which are given in Appendix I. Noted that $dA_i/d\omega$ and $d\phi_i/d\omega$ are given by (4), and

$$\frac{dA_i}{d\mathbf{h}_i}=\frac{1}{A_i}\left(\mathbf{c}_i\mathbf{c}_i^T+\mathbf{s}_i\mathbf{s}_i^T\right)\mathbf{h}_i=\frac{1}{A_i}\mathbf{P}_{mhi}\mathbf{h}_i \tag{10a}$$

$$\frac{d\phi_i}{d\mathbf{h}_i}=\frac{1}{A_i^2}\left(\mathbf{s}_i\mathbf{c}_i^T-\mathbf{c}_i\mathbf{s}_i^T\right)\mathbf{h}_i=\frac{1}{A_i^2}\mathbf{P}_{phi}\mathbf{h}_i \tag{10b}$$

$$\frac{d^2A_i}{d\omega d\mathbf{h}_i}=-\frac{1}{A_i}\frac{dA_i}{d\mathbf{h}_i}\frac{dA_i}{d\omega}+\frac{2}{A_i}\mathbf{P}_{m\omega i}\mathbf{h}_i \tag{10c}$$

$$\frac{d^2\phi_i}{d\omega d\mathbf{h}_i}=-\frac{2}{A_i}\frac{dA_i}{d\mathbf{h}_i}\frac{d\phi_i}{d\omega}+\frac{2}{A_i^2}\mathbf{P}_{p\omega i}\mathbf{h}_i \tag{10d}$$

for $i=1,6,7,3,4$.

As it can be seen from the above derivations for a two-stage FRM filter, we express the frequency response of each subfilter in magnitude and phase response, and then the frequency responses at the intermediate nodes as functions of the frequency responses of those subfilters involved, leading finally to the derivation of the required group delay and gradients of the two-stage FRM filter. Similar derivations can be carried out for a higher-stage FRM filter but more subfilters and intermediate nodes will be involved and the derivations are omitted here.

III. Design Methodology

The proposed design method takes into account both the magnitude error and the group delay error in the problem formulation. Based on the derivation described in the last section, a group delay constraint is formulated and incorporated into an overall optimization problem in the SOCP framework. The design methodology can be considered as a generalization of [6] by adding an additional group delay constraint.

A. Problem Formulation

Let $H_{des}(\omega)$ and $\tau_{des}(\omega)$ be the desired frequency response and passband group delay, and $H_f(\omega,\mathbf{x})$ and $\tau_f(\omega,\mathbf{x})$ be the designed frequency response and group delay, respectively, which depend on a real-valued design vector $\mathbf{x}$.

The classical minimax problem to approximate $H_{des}(\omega)$ can be stated as

$$\min_{\mathbf{x}}\left\{\max_{\omega\in\Omega_m}W_m(\omega)\left|H_f(\omega,\mathbf{x})-H_{des}(\omega)\right|\right\} \tag{11}$$

where Ω_m is the band of interest, which consists of the passband and stopband of the FRM filter, and $W_m(\omega)$ is a positive weighting function in magnitude.

Similarly, the minimax problem to approximate $\tau_{des}(\omega)$ can be stated as

$$\min_{\mathbf{x}}\left\{\max_{\omega\in\Omega_g}W_g(\omega)\left|\tau_f(\omega,\mathbf{x})-\tau_{des}(\omega)\right|\right\} \tag{12}$$

where Ω_g is the band of interest, which is the passband of the FRM filter, and $W_g(\omega)$ is a positive weighting function in group delay.

Denote the magnitude error, $\eta_m(\omega,\mathbf{x})=W_m(\omega)|H_f(\omega,\mathbf{x})-H_{des}(\omega)|$ for $\omega\in\Omega_m$ and the group delay error, $\eta_g(\omega,\mathbf{x})=W_g(\omega)|\tau_f(\omega,\mathbf{x})-\tau_{des}(\omega)|$ for

TABLE I
SIMULATION RESULTS FOR EXAMPLE 1

	β	$b_{\mathbf{v}}$	D_{rd}	δ_p(dB)	δ_s(dB)
Magnitude optimal FRM filter [6]	(na)	$1\times10^{-2}N_T$	0.0269	0.0401	-46.3261
New FRM filter 1	1×10^{-3}	$5\times10^{-4}N_T$	0.0225	0.0409	-46.0538
New FRM filter 2	2×10^{-3}	$5\times10^{-4}N_T$	0.0111	0.0411	-46.3233

$\omega\in\Omega_g$. The design of an FRM filter with reduced passband group delay can be formulated as

$$\min_{\mathbf{x}}\left\{\max_{\omega}\{\eta_m(\omega,\mathbf{x}),\beta\eta_g(\omega,\mathbf{x})\}\right\} \tag{13}$$

where β is a positive constant balancing the magnitude error and the group delay error. Note that "optimality" depends on the choice of β and the applications at hand. In general, larger β improves group delay performance at the costs of larger magnitude errors and vise versa.

B. Approximation Errors in Magnitude and Group Delay

The proposed design method for FRM filters with reduced group delays can be carried out in two steps. In the first step, a magnitude optimal FRM filter is obtained using the method of [6], which is then used as an initial solution for the overall optimization in the second step. Note that the phrase of "magnitude optimal FRM filter" is not very rigorous. It is used here solely for the convenience of distinguishing the FRM filters designed by the method of [6] using the criterion of magnitude error and those by our proposed method using the magnitude and group delay errors. Before addressing the overall optimization, we define the approximation errors in magnitude and group delay as follows.

Suppose we have a reasonable initial filter $\mathbf{x}_0$ to start optimization and we are now in the k-th iteration. For a nonlinear and smooth $H_f(\omega,\mathbf{x})$ at the vicinity of $\mathbf{x}_k$, i.e., at $\mathbf{x}_k+\mathbf{v}_k$ where $\mathbf{v}_k\in\mathcal{R}^{N_T\times1}$ is a small vector, we can write [6]

$$H_f(\omega,\mathbf{x}_k+\mathbf{v}_k)=H_f(\omega,\mathbf{x}_k)+\mathbf{g}_{m,k}^T(\omega)\mathbf{v}_k+o(\|\mathbf{v}_k\|)$$

where $o(\|\mathbf{v}_k\|)$ represents the summation of the higher order terms of the Taylor expansion, and $\mathbf{g}_{m,k}(\omega)$ is the gradient of $H_f(\omega,\mathbf{x})$ with respect to $\mathbf{x}$ given in (6) and with $\mathbf{x}=\mathbf{x}_k$. Hence, provided that $\|\mathbf{v}_k\|$ is small, the magnitude error becomes [6]

$$\begin{aligned}\eta_m(\omega,\mathbf{x}_k+\mathbf{v}_k)&\approx W_m(\omega)\\&\quad\times\left|\mathbf{g}_{m,k}^T(\omega)\mathbf{v}_k+[H_f(\omega,\mathbf{x}_k)-H_{des}(\omega)]\right|\\&=\left|\left(\mathbf{g}_{mr,k}^T(\omega)\mathbf{v}_k+e_{mr,k}(\omega)\right)\right.\\&\quad\left.+j\left(\mathbf{g}_{mi,k}^T(\omega)\mathbf{v}_k+e_{mi,k}(\omega)\right)\right|\end{aligned} \tag{14}$$

where $\mathbf{g}_{mr,k}$ and $\mathbf{g}_{mi,k}$ are the real and imaginary parts of $W_m(\omega)\mathbf{g}_{m,k}$, respectively, and

$$\begin{aligned}e_{mr,k}(\omega)&=W_m(\omega)\left[H_{fr}(\omega,\mathbf{x}_k)-H_{dr}(\omega)\right]\\e_{mi,k}(\omega)&=W_m(\omega)\left[H_{fi}(\omega,\mathbf{x}_k)-H_{di}(\omega)\right]\end{aligned}$$

with $H_{fr}(\omega,\mathbf{x}_k)$, $H_{dr}(\omega)$, $H_{fi}(\omega,\mathbf{x}_k)$ and $H_{di}(\omega)$ being the real and imaginary parts of $H_f(\omega,\mathbf{x}_k)$ and $H_{des}(\omega)$, respectively.

Similarly, the group delay error at the k-th iteration can be obtained through nonlinear approximation

$$\begin{aligned}\eta_g(\omega,\mathbf{x}_k+\mathbf{v}_k)&\approx W_g(\omega)\left|\mathbf{g}_{\tau,k}^T(\omega)\mathbf{v}_k+\tau_f(\omega,\mathbf{x}_k)-\tau_{des}(\omega)\right|\\&=\left|\mathbf{g}_{\tau r,k}^T(\omega)\mathbf{v}_k+e_{\tau r,k}(\omega)\right|\end{aligned} \tag{15}$$

where $\mathbf{g}_{\tau,k}(\omega)$ is the gradient of the group delay with respect to $\mathbf{x}$ given in (7) and with $\mathbf{x}=\mathbf{x}_k$, and

$$\begin{aligned}e_{\tau r,k}(\omega)&=W_g(\omega)\left(\tau_f(\omega,\mathbf{x}_k)-\tau_{des}(\omega)\right)\\\mathbf{g}_{\tau r,k}(\omega)&=W_g(\omega)\mathbf{g}_{\tau,k}(\omega).\end{aligned}$$

C. Overall Optimization

Using (14) and (15), it follows that an approximated solution of (13) at the k-th iteration can be obtained by solving the following problem

$$\min\quad\eta \tag{16a}$$

subject to :

$$\|\mathbf{G}_{m,k}(\omega)\mathbf{v}_k+\mathbf{e}_{m,k}(\omega)\|\le\eta\quad\text{for}\quad\omega\in\Omega_m \tag{16b}$$

$$\beta\left\|\left(\mathbf{g}_{\tau r,k}^T(\omega)\mathbf{v}_k+e_{\tau r,k}(\omega)\right)\right\|\le\eta\quad\text{for}\quad\omega\in\Omega_g \tag{16c}$$

$$\|\mathbf{v}_k\|\le b_{\mathbf{v}}. \tag{16d}$$

where $b_{\mathbf{v}}$ is a prescribed bound for $\mathbf{v}_k$,

$$\mathbf{G}_{m,k}(\omega)=\begin{bmatrix}\mathbf{g}_{mr,k}^T\\\mathbf{g}_{mi,k}^T\end{bmatrix}\quad\text{and}\quad\mathbf{e}_{m,k}(\omega)=\begin{bmatrix}e_{mr,k}(\omega)\\e_{mi,k}(\omega)\end{bmatrix}$$

Noted that (16c) is the group delay constraint.

If we treat the upper bound η as an additional design variable and define an augmented vector as $\mathbf{y}_k=[\eta\ \mathbf{v}_k^T]^T$, the objective function in (16a) can be expressed as $\eta=\mathbf{u}_1^T\mathbf{y}_k$ with $\mathbf{u}_1=[1\,0\,\ldots\,0]^T$. The constraints in (16b)–(16d) are actually quadratic cones that can be expressed in unit second-order cones under affine mapping. The constrained optimization problem defined by (16) can then be formulated as an SOCP problem by discretizing the frequency variable ω over a dense set of frequencies in the bands of interest. With m_K discrete frequency grids $\{\omega_1,\ldots,\omega_{m_K}\}\subset\Omega_m$ and g_K discrete frequency grids $\{\omega_1,\ldots,\omega_{g_K}\}\subset\Omega_g$, the k-th iteration problem in (16) becomes

$$\min\quad\mathbf{u}_1^T\mathbf{y}_k \tag{17a}$$

subject to :

$$\begin{bmatrix}1&\mathbf{0}^T\\\mathbf{0}&\mathbf{G}_{m,k}(\omega_l)\end{bmatrix}\mathbf{y}_k+\begin{bmatrix}0\\\mathbf{e}_{m,k}(\omega_l)\end{bmatrix}\in\mathcal{C}_l\qquad l=1,\ldots,m_K \tag{17b}$$

$$\begin{bmatrix}1&\mathbf{0}^T\\0&\beta\mathbf{g}_{\tau r,k}^T(\omega_i)\end{bmatrix}\mathbf{y}_k+\begin{bmatrix}0\\\beta e_{\tau r,k}(\omega_i)\end{bmatrix}\in\mathcal{C}_i,\qquad i=1,\ldots,g_K \tag{17c}$$

$$\begin{bmatrix}0&\mathbf{0}^T\\\mathbf{0}&\mathbf{I}\end{bmatrix}\mathbf{y}_k+\begin{bmatrix}b_{\mathbf{v}}\\\mathbf{0}\end{bmatrix}\in\mathcal{C}_p \tag{17d}$$

where $\mathbf{I}$ is an identity matrix of size N_T, and $\mathcal{C}_l$, $\mathcal{C}_i$ and $\mathcal{C}_p$ are the second-order cones in $\mathcal{R}^3$, $\mathcal{R}^2$ and $\mathcal{R}^{N_T+1}$, respectively. Note that there are $m_K + g_K + 1$ second-order cone constraints in (17), which can be solved efficiently by many interior-point methods, e.g. SeDuMi [22]. Having solved the problem in (17) for a minimizer $\mathbf{y}_k^* = [\eta_k^* \ (\mathbf{v}_k^*)^T]^T$, vector $\mathbf{v}_k^*$ is used to update $\mathbf{x}_k$ as $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{v}_k^*$. The iteration continues until change of $\|\mathbf{v}_k^*\|$ becomes insignificantly small compared to a prescribed tolerance.

With regards to the proposed design method, some points are highlighted in the following.

- Because of the nonlinearity of the frequency response and group delay with respect to an FRM filter coefficients, the proposed design method requires a good initial FRM filter, such as a magnitude optimal FRM filter using the method proposed in [6].
- Concerning the convergence of the proposed method, although a rigorous proof is presently not available, in our simulations when the method was applied to design a variety of FRM filters, it always converges. The number of updates depends on the initial design.
- Bound $b_{\mathbf{v}}$ in (16d) determines the deviation of $\mathbf{v}_k$ in successive update. We found the proposed design method is sensitive to $b_{\mathbf{v}}$, which is probably due to the incorporation of the group delay constraint. Based on our numerical simulations, we recommend it in the range between $2 \times 10^{-4} N_T$ and $1 \times 10^{-3} N_T$, which is somewhat smaller than the one suggested in [6].
- As a balancing factor between the magnitude and group delay errors, β is selected such that it is slightly larger than the ratio of the magnitude and group delay errors based on the initial design.

IV. Design Examples

In this section, the design of a two-stage FRM filter with cutoff frequencies $\omega_p = 0.6\pi, \omega_s = 0.61\pi$ and two different desirable passband group delays is considered. The design specifications are the same as those in [6] except for the desirable passband group delay in Example 2. To compare the performance of the designed filters, passband ripple, δ_p, stopband ripple, δ_s, both in decibels, and relative deviation in group delay, defined as $D_{rd} = \max_{\omega \in \Omega_g} |(\tau_f(\omega) - \tau_{des}(\omega))/\tau_{des}(\omega)|$, are adopted. For the overall optimization, the number of frequency grids is about $10N_T$ and relatively dense grid points are placed in the regions near the bandedges. In all the following related examples, we take $W_m(\omega) \equiv 1$ for $\omega \in \Omega_m$ and $W_g(\omega) \equiv 1$ for $\omega \in \Omega_g$.

Example 1, $\tau_{des}(\omega) = 207$: For comparison with the magnitude optimal FRM filter designed in [6], we constrain all the four masking filters to be of linear phase and adopt the same design parameters as in [6]: $M_1 = M_2 = 4$, $N_1 = 27$, $N_6 = 13$, $N_7 = 27$, $N_3 = 15$, $N_4 = 23$, $d_a = 9$, $d_{m1} = 11$ and $d_{m2} = 13$. Note that the dimension of the design vector $\mathbf{x}$ reduces to $N_T = N_1 + ((N_6+1)/2) + ((N_7+1)/2) + ((N_3+1)/2) + ((N_4+1)/2) = 68$ as all the four masking filters are of linear phases. Starting with the magnitude optimal FRM filter given in [6, p. 566] as an initial design and with $\beta = 1 \times 10^{-3}$, $b_{\mathbf{v}} = 5 \times 10^{-4} N_T$, the proposed design algorithm converges to an optimal FRM filter with some important measurements given in Table I. As it can be seen from the table, the passband and stopband errors of our new FRM filter are about the same as those in [6], but the relative deviation in group delay of the new FRM filter is reduced by 0.004, corresponding to about 16% reduction. Further reduction in group delay error can be achieved by increasing β. For example, when $\beta = 2 \times 10^{-3}$, the performance of the group delay of the new FRM filter improves significantly as the relative deviation in group delay is now reduced by 0.0158, corresponding to 58% reduction while the passband and stopband errors remain almost unchanged. Note that in general the improvement of group delay performance is at the cost of degradation of magnitude performance, as to be seen in the next example. We depict the performance of the magnitude optimal FRM filter [6] and the new FRM filter with $\beta = 2 \times 10^{-3}$ in Fig. 2. It should be pointed out that the relative deviation in group delay using the method by Lu and Hinamoto should be 0.0264 instead of 0.0132 given in [6, p. 566], the latter being arrived using an approximated method for evaluating D_{rd} [23].

Example 2, $\tau_{des}(\omega) = 164.8$: To meet the desired group delay which is now a non-integer, we have to relax the constraints of linear phase imposed on the masking filters. Although FRM filters with nonlinear phase masking filters were not addressed explicitly in [6], the feasibility of designing such filters was mentioned there. We first design a magnitude optimal FRM filter with linear phase masking filter (called Magnitude optimal FRM filter 1) using the method in [6] with the same design parameters $M_1, M_2, \ldots$ as in Example 1, except for $d_a = 6$ and $\tau_{des}(\omega) = 159$. Then we relax the constraints of the masking filters to obtain a magnitude optimal FRM filter with nonlinear phase masking filters (called magnitude optimal FRM filter 2). Since all four masking filters are of nonlinear phase, we now have $N_T = 105$. Using this magnitude optimal FRM filter as an initial design, an FRM filter with $\tau_{des}(\omega) = 164.8$ is approximated. Three new FRM filters are designed with $\beta = \{1 \times 10^{-3}, 2 \times 10^{-3}, 3 \times 10^{-3}\}$. Some important parameters of the designed filter are tabulated in Table II. From the table it can be seen that the group delay errors of the FRM filters designed by the proposed method are smaller than the ones of the FRM filters designed using the method in [6] although the magnitude errors increase slightly. Comparing the three FRM filters designed by the proposed method, it is seen that the group delay performance improves when β increases at the cost of slight degradation of the magnitude performance. Finally, the example has demonstrated that although the computational complexity of the FRM filters with nonlinear phase masking filter increases compared with FRM filters with linear phase masking filters, a distinctive advantage of relaxing the phase constraint on the masking filters is that arbitrary constant group delays can be approximated much more accurately.

V. Conclusion

In this paper, we have proposed a new method for designing optimal multistage FRM filters with reduced passband group delays. In order to reduce passband group delay error, detailed derivations of the group delay and its gradient with respect to the filter coefficients have been provided, and an explicit group delay constraint has been formulated and incorporated

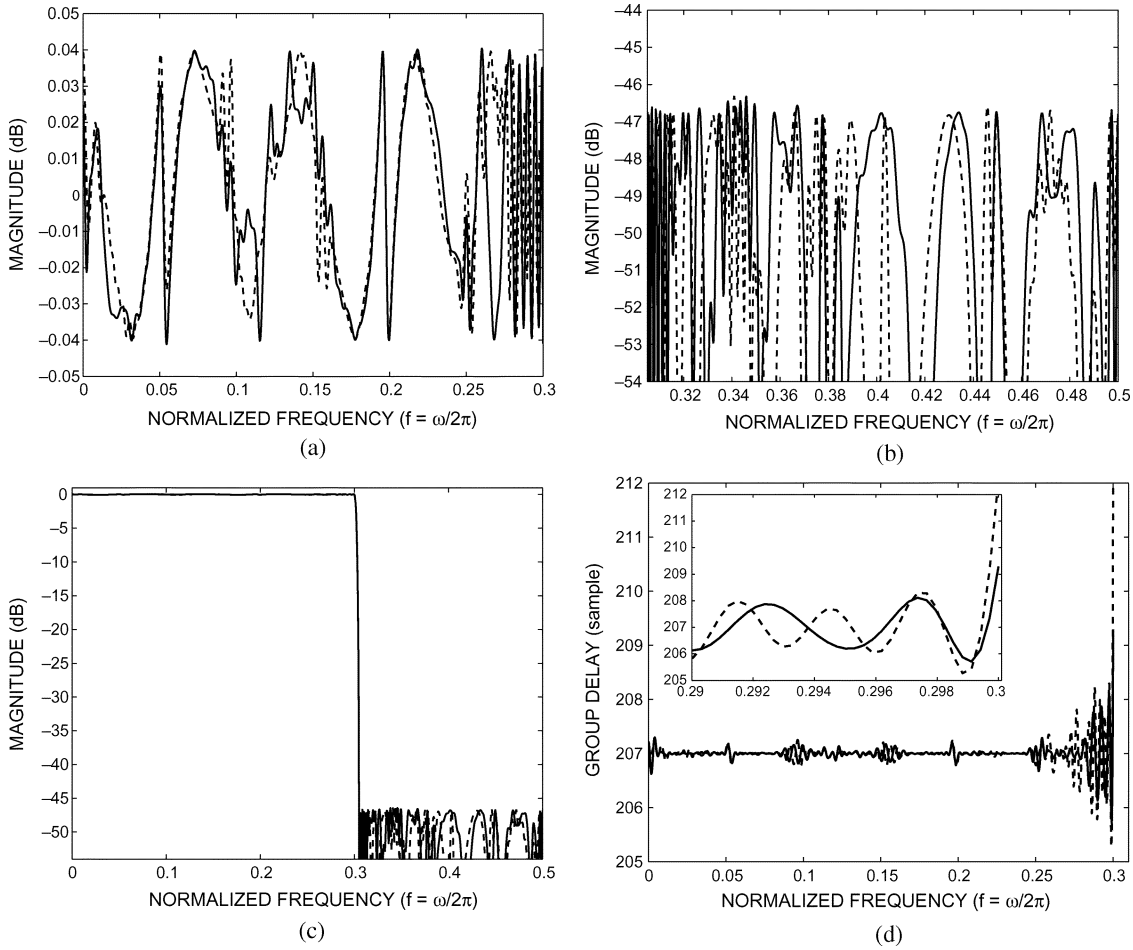

Fig. 2. Example 1. (a) Magnitude of passband ripples. (b) Magnitude of stopband ripples. (c) Magnitude responses. (d) Passband group delays of the magnitude optimal FRM filter (in broken lines) and the new FRM filter (in solid lines) with $\beta = 2 \times 10^{-3}$.

TABLE II
SIMULATION RESULTS FOR EXAMPLE 2

	$\tau_{des}(\omega)$	β	$b_{\mathbf{v}}$	D_{rd}	δ_p(dB)	δ_s(dB)
Magnitude optimal FRM filter 1	159	(na)	$4 \times 10^{-2} N_T$	0.0483	0.0664	-43.5515
Magnitude optimal FRM filter 2	159	(na)	$1 \times 10^{-4} N_T$	0.0399	0.0541	-43.9933
New FRM filter 1	164.8	1×10^{-3}	$1 \times 10^{-3} N_T$	0.0395	0.0522	-43.4928
New FRM filter 2	164.8	2×10^{-3}	$1 \times 10^{-3} N_T$	0.0323	0.0777	-41.8432
New FRM filter 3	164.8	3×10^{-3}	$1 \times 10^{-3} N_T$	0.0279	0.0711	-41.4762

in the design method. The effectiveness and flexibility of the proposed design method have been demonstrated by two related examples.

APPENDIX I
$d\tau_f(\omega)/d\mathbf{h}_i$ $(i = 1, 6, 7)$ FOR A TWO-STAGE FRM FILTER

A. Expressions of $d\mathcal{N}_f/d\mathbf{h}_i$ and $d\mathcal{D}_f/d\mathbf{h}_i$ for $i = 1, 6, 7$

$$\frac{d\mathcal{N}_f}{d\mathbf{h}_i} = \frac{dA_d}{d\mathbf{h}_i}\sin\phi_d + A_d\cos\phi_d\frac{d\phi_d}{d\mathbf{h}_i} + \frac{dA_e}{d\mathbf{h}_i}\sin\phi_e + A_e\cos\phi_e\frac{d\phi_e}{d\mathbf{h}_i}$$

$$\frac{d\mathcal{D}_f}{d\mathbf{h}_i} = \frac{dA_d}{d\mathbf{h}_i}\cos\phi_d - A_d\sin\phi_d\frac{d\phi_d}{d\mathbf{h}_i} + \frac{dA_e}{d\mathbf{h}_i}\cos\phi_e - A_e\sin\phi_e\frac{d\phi_e}{d\mathbf{h}_i}$$

$$\frac{dA_d}{d\mathbf{h}_i} = \frac{dA_c}{d\mathbf{h}_i}A_3,\ \frac{d\phi_d}{d\mathbf{h}_i} = \frac{d\phi_c}{d\mathbf{h}_i},\ \frac{dA_e}{d\mathbf{h}_i} = \frac{dA_5}{d\mathbf{h}_i}A_4,\ \frac{d\phi_e}{d\mathbf{h}_i} = \frac{d\phi_5}{d\mathbf{h}_i}$$

$$\frac{dA_5}{d\mathbf{h}_i} = \frac{1}{A_5}\left(\mathcal{N}_5\frac{d\mathcal{N}_5}{d\mathbf{h}_i} + \mathcal{D}_5\frac{d\mathcal{D}_5}{d\mathbf{h}_i}\right)$$

$$\frac{d\phi_5}{d\mathbf{h}_i} = \frac{1}{A_5^2}\left(\frac{d\mathcal{N}_5}{d\mathbf{h}_i}\mathcal{D}_5 - \mathcal{N}_5\frac{d\mathcal{D}_5}{d\mathbf{h}_i}\right)$$

$$\frac{d\mathcal{N}_5}{d\mathbf{h}_i} = -\frac{dA_c}{d\mathbf{h}_i}\sin\phi_c - A_c\cos\phi_c\frac{d\phi_c}{d\mathbf{h}_i}$$

$$\frac{d\mathcal{D}_5}{d\mathbf{h}_i} = -\frac{dA_c}{d\mathbf{h}_i}\cos\phi_c + A_c\sin\phi_c\frac{d\phi_c}{d\mathbf{h}_i}$$

$$\frac{dA_c}{d\mathbf{h}_i} = \frac{1}{A_c}\left(\mathcal{N}_c\frac{d\mathcal{N}_c}{d\mathbf{h}_i} + \mathcal{D}_c\frac{d\mathcal{D}_c}{d\mathbf{h}_i}\right)$$

$$\frac{d\phi_c}{d\mathbf{h}_i} = \frac{1}{A_c^2}\left(\frac{d\mathcal{N}_c}{d\mathbf{h}_i}\mathcal{D}_c - \mathcal{N}_c\frac{d\mathcal{D}_c}{d\mathbf{h}_i}\right).$$

B. Expressions of $d^2\mathcal{N}_f/d\omega d\mathbf{h}_i$ and $d^2\mathcal{D}_f/d\omega d\mathbf{h}_i$ for $i = 1, 6, 7$

$$\begin{aligned}\frac{d^2\mathcal{N}_f}{d\omega d\mathbf{h}_i} &= \frac{d^2A_d}{d\omega d\mathbf{h}_i}\sin\phi_d + \frac{dA_d}{d\omega}\cos\phi_d\frac{d\phi_d}{d\mathbf{h}_i} \\ &\quad + \frac{dA_d}{d\mathbf{h}_i}\cos\phi_d\frac{d\phi_d}{d\omega} - A_d\sin\phi_d\frac{d\phi_d}{d\mathbf{h}_i}\frac{d\phi_d}{d\omega} \\ &\quad + A_d\cos\phi_d\frac{d^2\phi_d}{d\omega d\mathbf{h}_i} + \frac{d^2A_e}{d\omega d\mathbf{h}_i}\sin\phi_e \\ &\quad + \frac{dA_e}{d\omega}\cos\phi_e\frac{d\phi_e}{d\mathbf{h}_i} + \frac{dA_e}{d\mathbf{h}_i}\cos\phi_e\frac{d\phi_e}{d\omega} \\ &\quad - A_e\sin\phi_e\frac{d\phi_e}{d\mathbf{h}_i}\frac{d\phi_e}{d\omega} + A_e\cos\phi_e\frac{d^2\phi_e}{d\omega d\mathbf{h}_i} \\ \frac{d^2\mathcal{D}_f}{d\omega d\mathbf{h}_i} &= \frac{d^2A_d}{d\omega d\mathbf{h}_i}\cos\phi_d - \frac{dA_d}{d\omega}\sin\phi_d\frac{d\phi_d}{d\mathbf{h}_i} \\ &\quad - \frac{dA_d}{d\mathbf{h}_i}\sin\phi_d\frac{d\phi_d}{d\omega} - A_d\cos\phi_d\frac{d\phi_d}{d\mathbf{h}_i}\frac{d\phi_d}{d\omega} \\ &\quad - A_d\sin\phi_d\frac{d^2\phi_d}{d\omega d\mathbf{h}_i} + \frac{d^2A_e}{d\omega d\mathbf{h}_i}\cos\phi_e \\ &\quad - \frac{dA_e}{d\omega}\sin\phi_e\frac{d\phi_e}{d\mathbf{h}_i} - \frac{dA_e}{d\mathbf{h}_i}\sin\phi_e\frac{d\phi_e}{d\omega} \\ &\quad - A_e\cos\phi_e\frac{d\phi_e}{d\mathbf{h}_i}\frac{d\phi_e}{d\omega} - A_e\sin\phi_e\frac{d^2\phi_e}{d\omega d\mathbf{h}_i} \\ \frac{d^2A_d}{d\omega d\mathbf{h}_i} &= \frac{d^2A_c}{d\omega d\mathbf{h}_i}A_3 + \frac{dA_c}{d\mathbf{h}_i}\frac{dA_3}{d\omega} \\ \frac{d^2\phi_d}{d\omega d\mathbf{h}_i} &= \frac{d^2\phi_c}{d\omega d\mathbf{h}_i} \\ \frac{d^2A_e}{d\omega d\mathbf{h}_i} &= \frac{d^2A_5}{d\omega d\mathbf{h}_i}A_4 + \frac{dA_5}{d\mathbf{h}_i}\frac{dA_4}{d\omega} \\ \frac{d^2\phi_e}{d\omega d\mathbf{h}_i} &= \frac{d^2\phi_5}{d\omega d\mathbf{h}_i} \\ \frac{d^2A_5}{d\omega d\mathbf{h}_i} &= \frac{-1}{A_5}\frac{dA_5}{d\mathbf{h}_i}\frac{dA_5}{d\omega} + \frac{1}{A_5}\left(\frac{d\mathcal{N}_5}{d\mathbf{h}_i}\frac{d\mathcal{N}_5}{d\omega} + \mathcal{N}_5\frac{d^2\mathcal{N}_5}{d\omega d\mathbf{h}_i}\right. \\ &\quad \left. + \frac{d\mathcal{D}_5}{d\mathbf{h}_i}\frac{d\mathcal{D}_5}{d\omega} + \mathcal{D}_5\frac{d^2\mathcal{D}_5}{d\omega d\mathbf{h}_i}\right) \\ \frac{d^2\phi_5}{d\omega d\mathbf{h}_i} &= \frac{-2}{A_5}\frac{dA_5}{d\mathbf{h}_i}\frac{d\phi_5}{d\omega} + \frac{1}{A_5^2}\left(\frac{d^2\mathcal{N}_5}{d\omega d\mathbf{h}_i}\mathcal{D}_5 + \frac{d\mathcal{N}_5}{d\omega}\frac{d\mathcal{D}_5}{d\mathbf{h}_i}\right. \\ &\quad \left. - \frac{d\mathcal{N}_5}{d\mathbf{h}_i}\frac{d\mathcal{D}_5}{d\omega} - \mathcal{N}_5\frac{d^2\mathcal{D}_5}{d\omega d\mathbf{h}_i}\right) \\ \frac{d^2\mathcal{N}_5}{d\omega d\mathbf{h}_i} &= -\frac{d^2A_c}{d\omega d\mathbf{h}_i}\sin\phi_c - \frac{dA_c}{d\omega}\cos\phi_c\frac{d\phi_c}{d\mathbf{h}_i} \\ &\quad - \frac{dA_c}{d\mathbf{h}_i}\cos\phi_c\frac{d\phi_c}{d\omega} + A_c\sin\phi_c\frac{d\phi_c}{d\mathbf{h}_i}\frac{d\phi_c}{d\omega} \\ &\quad - A_c\cos\phi_c\frac{d^2\phi_c}{d\omega d\mathbf{h}_i} \\ \frac{d^2\mathcal{D}_5}{d\omega d\mathbf{h}_i} &= -\frac{d^2A_c}{d\omega d\mathbf{h}_i}\cos\phi_c + \frac{dA_c}{d\omega}\sin\phi_c\frac{d\phi_c}{d\mathbf{h}_i} \\ &\quad + \frac{dA_c}{d\mathbf{h}_i}\sin\phi_c\frac{d\phi_c}{d\omega} + A_c\cos\phi_c\frac{d\phi_c}{d\mathbf{h}_i}\frac{d\phi_c}{d\omega} \\ &\quad + A_c\sin\phi_c\frac{d^2\phi_c}{d\omega d\mathbf{h}_i} \\ \frac{d^2A_c}{d\omega d\mathbf{h}_i} &= -\frac{1}{A_c}\frac{dA_c}{d\mathbf{h}_i}\frac{dA_c}{d\omega} + \frac{1}{A_c}\left(\frac{d\mathcal{N}_c}{d\mathbf{h}_i}\frac{d\mathcal{N}_c}{d\omega} + \mathcal{N}_c\frac{d^2\mathcal{N}_c}{d\omega d\mathbf{h}_i}\right. \\ &\quad \left. + \frac{d\mathcal{D}_c}{d\mathbf{h}_i}\frac{d\mathcal{D}_c}{d\omega} + \mathcal{D}_c\frac{d^2\mathcal{D}_c}{d\omega d\mathbf{h}_i}\right) \\ \frac{d^2\phi_c}{d\omega d\mathbf{h}_i} &= \frac{-2}{A_c}\frac{dA_c}{d\mathbf{h}_i}\frac{d\phi_c}{d\omega} + \frac{1}{A_c^2}\left(\frac{d^2\mathcal{N}_c}{d\omega d\mathbf{h}_i}\mathcal{D}_c + \frac{d\mathcal{N}_c}{d\omega}\frac{d\mathcal{D}_c}{d\mathbf{h}_i}\right. \\ &\quad \left. - \frac{d\mathcal{N}_c}{d\mathbf{h}_i}\frac{d\mathcal{D}_c}{d\omega} - \mathcal{N}_c\frac{d^2\mathcal{D}_c}{d\omega d\mathbf{h}_i}\right).\end{aligned}$$

C. Expressions of $d\tau_f(\omega)/d\mathbf{h}_i$ for $i = 6, 7$

$$\begin{aligned}\frac{d\mathcal{N}_c}{d\mathbf{h}_i} &= \frac{dA_l}{d\mathbf{h}_i}\sin\phi_l + A_l\cos\phi_l\frac{d\phi_l}{d\mathbf{h}_i} \\ \frac{dA_l}{d\mathbf{h}_i} &= A_q\frac{dA_i}{d\mathbf{h}_i} \\ \frac{d\mathcal{D}_c}{d\mathbf{h}_i} &= \frac{dA_l}{d\mathbf{h}_i}\cos\phi_l - A_l\sin\phi_l\frac{d\phi_l}{d\mathbf{h}_i} \\ \frac{d\phi_l}{d\mathbf{h}_i} &= \frac{d\phi_i}{d\mathbf{h}_i} \\ \frac{d^2\mathcal{N}_c}{d\omega d\mathbf{h}_i} &= \frac{d^2A_l}{d\omega d\mathbf{h}_i}\sin\phi_l + \frac{dA_l}{d\omega}\cos\phi_l\frac{d\phi_l}{d\mathbf{h}_i} + \frac{dA_l}{d\mathbf{h}_i}\cos\phi_l\frac{d\phi_l}{d\omega} \\ &\quad - A_l\sin\phi_l\frac{d\phi_l}{d\mathbf{h}_i}\frac{d\phi_l}{d\omega} + A_l\cos\phi_l\frac{d^2\phi_l}{d\omega d\mathbf{h}_i} \\ \frac{d^2\mathcal{D}_c}{d\omega d\mathbf{h}_i} &= \frac{d^2A_l}{d\omega d\mathbf{h}_i}\cos\phi_l - \frac{dA_l}{d\omega}\sin\phi_l\frac{d\phi_l}{d\mathbf{h}_i} - \frac{dA_l}{d\mathbf{h}_i}\sin\phi_l\frac{d\phi_l}{d\omega} \\ &\quad - A_l\cos\phi_l\frac{d\phi_l}{d\mathbf{h}_i}\frac{d\phi_l}{d\omega} - A_l\sin\phi_l\frac{d^2\phi_l}{d\omega d\mathbf{h}_i} \\ \frac{d^2A_l}{d\omega d\mathbf{h}_i} &= \frac{dA_q}{d\omega}\frac{dA_i}{d\mathbf{h}_i} + A_q\frac{d^2A_i}{d\omega d\mathbf{h}_i} \\ \frac{d^2\phi_l}{d\omega d\mathbf{h}_i} &= \frac{d^2\phi_i}{d\omega d\mathbf{h}_i}, \quad \text{for } (i,l,q) = (6,a,1) \text{ or } (i,l,q) = (7,b,2).\end{aligned}$$

D. Expressions of $d\tau_f(\omega)/d\mathbf{h}_1$

$$\begin{aligned}\frac{d\mathcal{N}_c}{d\mathbf{h}_1} &= \frac{dA_a}{d\mathbf{h}_1}\sin\phi_a + A_a\cos\phi_a\frac{d\phi_a}{d\mathbf{h}_1} + \frac{dA_b}{d\mathbf{h}_1}\sin\phi_b \\ &\quad + A_b\cos\phi_b\frac{d\phi_b}{d\mathbf{h}_1} \\ \frac{d\mathcal{D}_c}{d\mathbf{h}_1} &= \frac{dA_a}{d\mathbf{h}_1}\cos\phi_a - A_a\sin\phi_a\frac{d\phi_a}{d\mathbf{h}_1} + \frac{dA_b}{d\mathbf{h}_1}\cos\phi_b \\ &\quad - A_b\sin\phi_b\frac{d\phi_b}{d\mathbf{h}_1} \\ \frac{dA_a}{d\mathbf{h}_1} &= A_b\frac{dA_1}{d\mathbf{h}_1} \\ \frac{d\phi_a}{d\mathbf{h}_1} &= \frac{d\phi_1}{d\mathbf{h}_1} \\ \frac{dA_b}{d\mathbf{h}_1} &= A_7\frac{dA_2}{d\mathbf{h}_1} \\ \frac{d\phi_b}{d\mathbf{h}_1} &= \frac{d\phi_2}{d\mathbf{h}_1} \\ \frac{dA_2}{d\mathbf{h}_1} &= \frac{1}{A_2}\left(\mathcal{N}_2\frac{d\mathcal{N}_2}{d\mathbf{h}_1} + \mathcal{D}_2\frac{d\mathcal{D}_2}{d\mathbf{h}_1}\right) \\ \frac{d\phi_2}{d\mathbf{h}_1} &= \frac{1}{A_2^2}\left(\frac{d\mathcal{N}_2}{d\mathbf{h}_1}\mathcal{D}_2 - \mathcal{N}_2\frac{d\mathcal{D}_2}{d\mathbf{h}_1}\right) \\ \frac{d\mathcal{N}_2}{d\mathbf{h}_1} &= -\frac{dA_1}{d\mathbf{h}_1}\sin\phi_1 - A_1\cos\phi_1\frac{d\phi_1}{d\mathbf{h}_1} \\ \frac{d\mathcal{D}_2}{d\mathbf{h}_1} &= -\frac{dA_1}{d\mathbf{h}_1}\cos\phi_1 + A_1\sin\phi_1\frac{d\phi_1}{d\mathbf{h}_1}\end{aligned}$$

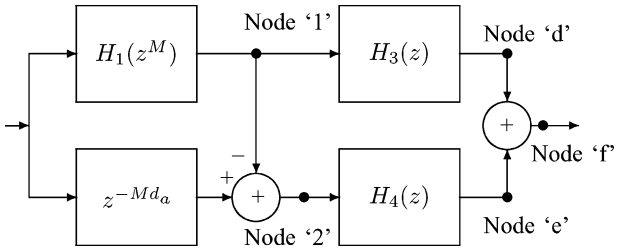

Fig. 3. Basic FRM filter structure.

$$\frac{d^2\mathcal{N}_c}{d\omega d\mathbf{h}_1} = \frac{d^2A_a}{d\omega d\mathbf{h}_1}\sin\phi_a + \frac{dA_a}{d\omega}\cos\phi_a\frac{d\phi_a}{d\mathbf{h}_1} + \frac{dA_a}{d\mathbf{h}_1}\cos\phi_a\frac{d\phi_a}{d\omega} - A_a\sin\phi_a\frac{d\phi_a}{d\mathbf{h}_1}\frac{d\phi_a}{d\omega} + A_a\cos\phi_a\frac{d^2\phi_a}{d\omega d\mathbf{h}_1} + \frac{d^2A_b}{d\omega d\mathbf{h}_1}\sin\phi_b + \frac{dA_b}{d\omega}\cos\phi_b\frac{d\phi_b}{d\mathbf{h}_1} + \frac{dA_b}{d\mathbf{h}_1}\cos\phi_b\frac{d\phi_b}{d\omega} - A_b\sin\phi_b\frac{d\phi_b}{d\mathbf{h}_1}\frac{d\phi_b}{d\omega} + A_b\cos\phi_b\frac{d^2\phi_b}{d\omega d\mathbf{h}_1}$$

$$\frac{d^2\mathcal{D}_c}{d\omega d\mathbf{h}_1} = \frac{d^2A_a}{d\omega d\mathbf{h}_1}\cos\phi_a - \frac{dA_a}{d\omega}\sin\phi_a\frac{d\phi_a}{d\mathbf{h}_1} - \frac{dA_a}{d\mathbf{h}_1}\sin\phi_a\frac{d\phi_a}{d\omega} - A_a\cos\phi_a\frac{d\phi_a}{d\mathbf{h}_1}\frac{d\phi_a}{d\omega} - A_a\sin\phi_a\frac{d^2\phi_a}{d\omega d\mathbf{h}_1} + \frac{d^2A_b}{d\omega d\mathbf{h}_1}\cos\phi_b - \frac{dA_b}{d\omega}\sin\phi_b\frac{d\phi_b}{d\mathbf{h}_1} - \frac{dA_b}{d\mathbf{h}_1}\sin\phi_b\frac{d\phi_b}{d\omega} - A_b\cos\phi_b\frac{d\phi_b}{d\mathbf{h}_1}\frac{d\phi_b}{d\omega} - A_b\sin\phi_b\frac{d^2\phi_b}{d\omega d\mathbf{h}_1}$$

$$\frac{d^2A_b}{d\omega d\mathbf{h}_1} = \frac{d^2A_2}{d\omega d\mathbf{h}_1}A_7 + \frac{dA_2}{d\mathbf{h}_1}\frac{dA_7}{d\omega}$$

$$\frac{d^2\phi_b}{d\omega d\mathbf{h}_1} = \frac{d^2\phi_2}{d\omega d\mathbf{h}_1}$$

$$\frac{d^2A_a}{d\omega d\mathbf{h}_1} = \frac{d^2A_1}{d\omega d\mathbf{h}_1}A_6 + \frac{dA_1}{d\mathbf{h}_1}\frac{dA_6}{d\omega}$$

$$\frac{d^2\phi_a}{d\omega d\mathbf{h}_1} = \frac{d^2\phi_1}{d\omega d\mathbf{h}_1}$$

$$\frac{d^2A_2}{d\omega d\mathbf{h}_1} = -\frac{1}{A_2}\frac{dA_2}{d\mathbf{h}_1}\frac{dA_2}{d\omega} + \frac{1}{A_2}\left(\frac{d\mathcal{N}_2}{d\mathbf{h}_1}\frac{d\mathcal{N}_2}{d\omega} + \mathcal{N}_2\frac{d^2\mathcal{N}_2}{d\omega d\mathbf{h}_1} + \frac{d\mathcal{D}_2}{d\mathbf{h}_1}\frac{d\mathcal{D}_2}{d\omega} + \mathcal{D}_2\frac{d^2\mathcal{D}_2}{d\omega d\mathbf{h}_1}\right)$$

$$\frac{d^2\phi_2}{d\omega d\mathbf{h}_1} = -\frac{2}{A_2}\frac{dA_2}{d\mathbf{h}_1}\frac{d\phi_2}{d\omega} + \frac{2}{A_2^2}\left(\frac{d^2\mathcal{N}_2}{d\omega d\mathbf{h}_1}\mathcal{D}_2 + \frac{d\mathcal{N}_2}{d\omega}\frac{d\mathcal{N}_2}{d\mathbf{h}_1} - \frac{d^2\mathcal{D}_2}{d\omega d\mathbf{h}_1}\mathcal{N}_2 + \frac{d\mathcal{D}_2}{d\omega}\frac{d\mathcal{N}_2}{d\mathbf{h}_1}\right)$$

$$\frac{d^2\mathcal{N}_2}{d\omega d\mathbf{h}_1} = -\frac{d^2A_1}{d\omega d\mathbf{h}_1}\sin\phi_1 - \frac{dA_1}{d\omega}\cos\phi_1\frac{d\phi_1}{d\mathbf{h}_1} - \frac{dA_1}{d\mathbf{h}_1}\cos\phi_1\frac{d\phi_1}{d\omega} + A_1\sin\phi_1\frac{d\phi_1}{d\mathbf{h}_1}\frac{d\phi_1}{d\omega} - A_1\cos\phi_1\frac{d^2\phi_1}{d\omega d\mathbf{h}_1}$$

$$\frac{d^2\mathcal{D}_2}{d\omega d\mathbf{h}_1} = -\frac{d^2A_1}{d\omega d\mathbf{h}_1}\cos\phi_1 + \frac{dA_1}{d\omega}\sin\phi_1\frac{d\phi_1}{d\mathbf{h}_1} + \frac{dA_1}{d\mathbf{h}_1}\sin\phi_1\frac{d\phi_1}{d\omega} + A_1\cos\phi_1\frac{d\phi_1}{d\mathbf{h}_1}\frac{d\phi_1}{d\omega} + A_1\sin\phi_1\frac{d^2\phi_1}{d\omega d\mathbf{h}_1}.$$

APPENDIX II
GROUP DELAY AND GRADIENTS FOR BASIC FRM FILTERS

As shown in Fig. 3, a basic FRM filter consists of a prototype filter $H_1(z)$ and a pair of masking filters $\{H_3(z), H_4(z)\}$. The vectors of the coefficients of the subfilter are $\mathbf{h}_1$, $\mathbf{h}_3$ and $\mathbf{h}_4$ of lengths N_1, N_3 and N_4, respectively, and that M is a scaling factor.

A. Group Delay for Basic FRM Filters

Denote $N_m = \max\{N_3, N_4\}$ and d_m the group delay of the masking filters.

Let $A_1e^{j\phi_1}$ (at node "1"), $A_3e^{j\phi_3}$ and $A_4e^{j\phi_4}$ be the frequency responses of $H_1(z^M)$, $H_3(z)$ and $H_4(z)$, respectively. We have $A_i = \sqrt{(\mathbf{h}_i^T\mathbf{c}_i)^2 + (\mathbf{h}_i^T/\mathbf{s}_i)^2}$ and $\phi_i = \arctan 2(\mathbf{h}_i^T\mathbf{s}_i, \mathbf{h}_i^T\mathbf{c}_i)$ for $i = 1, 3, 4$, where

$$\mathbf{c}_1 = [1 \;\; \cos M\omega \;\; \cos 2M\omega \;\; \ldots \;\; \cos(N_1-1)M\omega]^T$$

$$\mathbf{s}_1 = [0 \;\; -\sin M\omega \;\; -\sin 2M\omega \;\; \ldots \;\; -\sin(N_1-1)M\omega]^T$$

$$\mathbf{c}_l = \left[\cos\frac{N_m-N_l}{2}\omega \;\; \ldots \;\; \cos\frac{N_m-1}{2}\omega \;\; \ldots \;\; \cos\frac{N_m+N_l-2}{2}\omega\right]^T, \qquad l=3,4$$

$$\mathbf{s}_l = \left[-\sin\frac{N_m-N_l}{2}\omega \;\; \ldots \;\; -\sin\frac{N_m-1}{2}\omega \;\; \ldots \;\; -\sin\frac{N_m+N_l-2}{2}\omega\right]^T, \qquad l=3,4.$$

The frequency responses at nodes "2," "d," "e," and "f" in Fig. 3 can be obtained as

$$A_2e^{j\phi_2} = e^{-j\omega Md_a} - A_1e^{j\phi_1}$$
$$A_de^{j\phi_d} = A_1A_3e^{j(\phi_1+\phi_3)}$$
$$A_ee^{j\phi_e} = A_2A_4e^{j(\phi_2+\phi_4)}$$
$$A_fe^{j\phi_f} = A_de^{j\phi_d} + A_ee^{j\phi_e}.$$

Let $\mathcal{D}_2$, $\mathcal{D}_f$ and $\mathcal{N}_2$, $\mathcal{N}_f$ be the real and imaginary parts of $A_2e^{j\phi_2}$, and $A_fe^{j\phi_f}$, respectively. We have $\mathcal{N}_f = A_d\sin\phi_d + A_e\sin\phi_e$ and $\mathcal{D}_f = A_d\cos\phi_d + A_e\cos\phi_e$. For the group delay of the basic FRM filter given by (1), $d\mathcal{N}_f/d\omega$ and $d\mathcal{D}_f/d\omega$ as functions of $dA_d/d\omega$, $d\phi_d/d\omega$, $dA_e/d\omega$ and $d\phi_e/d\omega$ are given by (2). And $dA_d/d\omega$, $dA_e/d\omega$, $d\phi_d/d\omega$ and $d\phi_e/d\omega$ can be subsequently obtained as functions of $dA_i/d\omega$ and $d\phi_i/d\omega$ $(i = 1, 3, 4)$ given by

$$\frac{dA_d}{d\omega} = \frac{dA_1}{d\omega}A_3 + A_1\frac{dA_3}{d\omega}$$
$$\frac{d\phi_d}{d\omega} = \frac{d\phi_1}{d\omega} + \frac{d\phi_3}{d\omega}$$
$$\frac{dA_e}{d\omega} = \frac{dA_2}{d\omega}A_4 + A_2\frac{dA_4}{d\omega}$$
$$\frac{d\phi_e}{d\omega} = \frac{d\phi_2}{d\omega} + \frac{d\phi_4}{d\omega}$$

where $dA_i/d\omega$ and $d\phi_i/d\omega$ ($i = 1, 3, 4$) are given in (4).

B. Gradients for Basic FRM Filters

The frequency response of the basic FRM filter is

$$H_f(\omega) = H_1(M\omega)H_3(\omega) + \left(e^{-jMd_a\omega} - H_1(M\omega)\right) H_4(\omega)$$

When the design variables are put together as a design vector $\mathbf{x} = [\mathbf{h}_1^T\ \mathbf{h}_3^T\ \mathbf{h}_4^T]^T$ of dimension $N_T = N_1 + N_3 + N_4$, the gradient of the frequency response with respect to $\mathbf{x}$ is

$$\mathbf{g}_m(\omega) = \begin{bmatrix} \frac{dH_f(\omega)}{d\mathbf{h}_1} \\ \frac{dH_f(\omega)}{d\mathbf{h}_3} \\ \frac{dH_f(\omega)}{d\mathbf{h}_4} \end{bmatrix} = \begin{bmatrix} (H_3(\omega) - H_4(\omega))\,\mathbf{e}_1 \\ H_1(M\omega)\mathbf{e}_3 \\ \left(e^{-jMd_a\omega} - H_1(M\omega)\right)\mathbf{e}_4 \end{bmatrix}$$

where

$$\mathbf{e}_1 = \begin{bmatrix} 1 & e^{-jM\omega} & e^{-j2M\omega} & \dots & e^{-j(N_1-1)M\omega} \end{bmatrix}^T$$
$$\mathbf{e}_l = \begin{bmatrix} e^{-j\frac{N_m-N_l}{2}\omega} & e^{-j\frac{N_m-N_l+2}{2}\omega} & \dots & e^{-j\frac{N_m+N_l-2}{2}\omega} \end{bmatrix}^T, \quad l = 3, 4.$$

Using the group delay expressions derived above, the gradient of the group delay with respect to $\mathbf{x}$ is

$$\mathbf{g}_\tau(\omega) = \begin{bmatrix} \frac{d\tau_f(\omega)}{d\mathbf{h}_1} \\ \frac{d\tau_f(\omega)}{d\mathbf{h}_3} \\ \frac{d\tau_f(\omega)}{d\mathbf{h}_4} \end{bmatrix}$$

whose components, the gradient of $\tau_f(\omega)$ with respect to $\mathbf{h}_1$, $\mathbf{h}_3$ and $\mathbf{h}_4$ are given in (8). $d\mathcal{N}_f/d\mathbf{h}_i$, $d\mathcal{D}_f/d\mathbf{h}_i$, $d^2\mathcal{N}_f/d\omega d\mathbf{h}_i$ and $d^2\mathcal{D}_f/d\omega d\mathbf{h}_i$ for $i = \{3, 4\}$ are given in (9) for $(i, l, q) = (3, 1, d)$ or $(i, l, q) = (4, 2, e)$, while those for $i = 1$ are given below

$$\begin{aligned}
\frac{d\mathcal{N}_f}{d\mathbf{h}_1} &= \frac{dA_d}{d\mathbf{h}_1}\sin\phi_d + A_d\cos\phi_d\frac{d\phi_d}{d\mathbf{h}_1} + \frac{dA_e}{d\mathbf{h}_1}\sin\phi_e \\
&\quad + A_e\cos\phi_e\frac{d\phi_e}{d\mathbf{h}_1} \\
\frac{d\mathcal{D}_f}{d\mathbf{h}_1} &= \frac{dA_d}{d\mathbf{h}_1}\cos\phi_d - A_d\sin\phi_d\frac{d\phi_d}{d\mathbf{h}_1} + \frac{dA_e}{d\mathbf{h}_1}\cos\phi_e \\
&\quad - A_e\sin\phi_e\frac{d\phi_e}{d\mathbf{h}_1} \\
\frac{dA_d}{d\mathbf{h}_1} &= A_3\frac{dA_1}{d\mathbf{h}_1}, \quad \frac{dA_e}{d\mathbf{h}_1} = A_4\frac{dA_2}{d\mathbf{h}_1} \\
\frac{d\phi_d}{d\mathbf{h}_1} &= \frac{d\phi_1}{d\mathbf{h}_1}, \quad \frac{d\phi_e}{d\mathbf{h}_1} = \frac{d\phi_2}{d\mathbf{h}_1} \\
\frac{dA_2}{d\mathbf{h}_1} &= \frac{1}{A_2}\left(\mathcal{N}_2\frac{d\mathcal{N}_2}{d\mathbf{h}_1} + \mathcal{D}_2\frac{d\mathcal{D}_2}{d\mathbf{h}_1}\right) \\
\frac{d\phi_2}{d\mathbf{h}_1} &= \frac{1}{A_2^2}\left(\frac{d\mathcal{N}_2}{d\mathbf{h}_1}\mathcal{D}_2 - \mathcal{N}_2\frac{d\mathcal{D}_2}{d\mathbf{h}_1}\right) \\
\frac{d\mathcal{N}_2}{d\mathbf{h}_1} &= -\frac{dA_1}{d\mathbf{h}_1}\sin\phi_1 - A_1\cos\phi_1\frac{d\phi_1}{d\mathbf{h}_1} \\
\frac{d\mathcal{D}_2}{d\mathbf{h}_1} &= -\frac{dA_1}{d\mathbf{h}_1}\cos\phi_1 + A_1\sin\phi_1\frac{d\phi_1}{d\mathbf{h}_1}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2\mathcal{N}_f}{d\omega d\mathbf{h}_1} &= \frac{d^2A_d}{d\omega d\mathbf{h}_1}\sin\phi_d + \frac{dA_d}{d\omega}\cos\phi_d\frac{d\phi_d}{d\mathbf{h}_1} \\
&\quad + \frac{dA_d}{d\mathbf{h}_1}\cos\phi_d\frac{d\phi_d}{d\omega} - A_d\sin\phi_d\frac{d\phi_d}{d\mathbf{h}_1}\frac{d\phi_d}{d\omega} \\
&\quad + A_d\cos\phi_d\frac{d^2\phi_d}{d\omega d\mathbf{h}_1} + \frac{d^2A_e}{d\omega d\mathbf{h}_1}\sin\phi_e \\
&\quad + \frac{dA_e}{d\omega}\cos\phi_e\frac{d\phi_e}{d\mathbf{h}_1} + \frac{dA_e}{d\mathbf{h}_1}\cos\phi_e\frac{d\phi_e}{d\omega} \\
&\quad - A_e\sin\phi_e\frac{d\phi_e}{d\mathbf{h}_1}\frac{d\phi_e}{d\omega} + A_e\cos\phi_e\frac{d^2\phi_e}{d\omega d\mathbf{h}_1} \\
\frac{d^2\mathcal{D}_f}{d\omega d\mathbf{h}_1} &= \frac{d^2A_d}{d\omega d\mathbf{h}_1}\cos\phi_d - \frac{dA_d}{d\omega}\sin\phi_d\frac{d\phi_d}{d\mathbf{h}_1} \\
&\quad - \frac{dA_d}{d\mathbf{h}_1}\sin\phi_d\frac{d\phi_d}{d\omega} - A_d\cos\phi_d\frac{d\phi_d}{d\mathbf{h}_1}\frac{d\phi_d}{d\omega} \\
&\quad - A_d\sin\phi_d\frac{d^2\phi_d}{d\omega d\mathbf{h}_1} + \frac{d^2A_e}{d\omega d\mathbf{h}_1}\cos\phi_e \\
&\quad - \frac{dA_e}{d\omega}\sin\phi_e\frac{d\phi_e}{d\mathbf{h}_1} - \frac{dA_e}{d\mathbf{h}_1}\sin\phi_e\frac{d\phi_e}{d\omega} \\
&\quad - A_e\cos\phi_e\frac{d\phi_e}{d\mathbf{h}_1}\frac{d\phi_e}{d\omega} - A_e\sin\phi_e\frac{d^2\phi_e}{d\omega d\mathbf{h}_1} \\
\frac{d^2A_e}{d\omega d\mathbf{h}_1} &= \frac{d^2A_2}{d\omega d\mathbf{h}_1}A_4 + \frac{dA_2}{d\mathbf{h}_1}\frac{dA_4}{d\omega}, \quad \frac{d^2\phi_e}{d\omega d\mathbf{h}_1} = \frac{d^2\phi_2}{d\omega d\mathbf{h}_1} \\
\frac{d^2A_d}{d\omega d\mathbf{h}_1} &= \frac{d^2A_1}{d\omega d\mathbf{h}_1}A_3 + \frac{dA_1}{d\mathbf{h}_1}\frac{dA_3}{d\omega}, \quad \frac{d^2\phi_d}{d\omega d\mathbf{h}_1} = \frac{d^2\phi_1}{d\omega d\mathbf{h}_1} \\
\frac{d^2A_2}{d\omega d\mathbf{h}_1} &= -\frac{1}{A_2}\frac{dA_2}{d\mathbf{h}_1}\frac{dA_2}{d\omega} + \frac{1}{A_2}\left(\frac{d\mathcal{N}_2}{d\mathbf{h}_1}\frac{d\mathcal{N}_2}{d\omega} + \mathcal{N}_2\frac{d^2\mathcal{N}_2}{d\omega d\mathbf{h}_1}\right. \\
&\quad \left. + \frac{d\mathcal{D}_2}{d\mathbf{h}_1}\frac{d\mathcal{D}_2}{d\omega} + \mathcal{D}_2\frac{d^2\mathcal{D}_2}{d\omega d\mathbf{h}_1}\right) \\
\frac{d^2\phi_2}{d\omega d\mathbf{h}_1} &= -\frac{2}{A_2}\frac{dA_2}{d\mathbf{h}_1}\frac{d\phi_2}{d\omega} + \frac{2}{A_2^2}\left(\frac{d^2\mathcal{N}_2}{d\omega d\mathbf{h}_1}\mathcal{D}_2 + \frac{d\mathcal{N}_2}{d\omega}\frac{d\mathcal{N}_2}{d\mathbf{h}_1}\right. \\
&\quad \left. - \frac{d^2\mathcal{D}_2}{d\omega d\mathbf{h}_1}\mathcal{N}_2 + \frac{d\mathcal{D}_2}{d\omega}\frac{d\mathcal{N}_2}{d\mathbf{h}_1}\right) \\
\frac{d^2\mathcal{N}_2}{d\omega d\mathbf{h}_1} &= -\frac{d^2A_1}{d\omega d\mathbf{h}_1}\sin\phi_1 - \frac{dA_1}{d\omega}\cos\phi_1\frac{d\phi_1}{d\mathbf{h}_1} \\
&\quad - \frac{dA_1}{d\mathbf{h}_1}\cos\phi_1\frac{d\phi_1}{d\omega} + A_1\sin\phi_1\frac{d\phi_1}{d\mathbf{h}_1}\frac{d\phi_1}{d\omega} \\
&\quad - A_1\cos\phi_1\frac{d^2\phi_1}{d\omega d\mathbf{h}_1} \\
\frac{d^2\mathcal{D}_2}{d\omega d\mathbf{h}_1} &= -\frac{d^2A_1}{d\omega d\mathbf{h}_1}\cos\phi_1 + \frac{dA_1}{d\omega}\sin\phi_1\frac{d\phi_1}{d\mathbf{h}_1} \\
&\quad + \frac{dA_1}{d\mathbf{h}_1}\sin\phi_1\frac{d\phi_1}{d\omega} + A_1\cos\phi_1\frac{d\phi_1}{d\mathbf{h}_1}\frac{d\phi_1}{d\omega} \\
&\quad + A_1\sin\phi_1\frac{d^2\phi_1}{d\omega d\mathbf{h}_1}.
\end{aligned}$$

Note that $dA_i/d\mathbf{h}_i$, $d\phi_i/d\mathbf{h}_i$, $d^2A_i/d\omega d\mathbf{h}_i$ and $d^2\phi_i/d\omega d\mathbf{h}_i$ ($i = 1, 3, 4$) are given by (10).

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