FRM Filter Design with Group Delay Constraint Using Second-Order Cone Programming

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***Abstract*— This paper presents an improved method for the design of frequency-response masking (FRM) FIR filters with reduced passband group delays. Expressions of the group delay and its gradient for an FRM filter are derived. To directly and effectively control the group delay error, a group delay constraint is formulated and incorporated into the overall optimization. The effectiveness and flexibility of the proposed design method are demonstrated by an example, which shows that significant reduction in group delay error could be achieved at the cost of slight increase in magnitude error.**

I. Introduction

The design of linear phase frequency-response masking (FRM) filters has attracted much attention in the past two decades since the FRM technique [1] is well known to be very efficient in the implementation of digital filters with sharp transition bands. However, the effective length of a linear phase FRM FIR filter is slightly longer than that of a direct-form FIR filter designed by conventional methods with the same approximation accuracy, leading to slightly larger overall group delay. As a result, linear phase FRM FIR filters with sharp transition bands have very large group delays, which are undesirable in many applications. This has motivated some recent interest in designing FRM FIR filters with reduced passband group delays [2]–[5]. In these methods, the prescribed group delays are indirectly achieved by minimizing the combined magnitude and phase errors, i.e., the complex magnitude errors. Without any direct constraint in group delay, it is not easy to control the group delay error, leading to relatively large group delay error, noticeably towards the bandedges.

In this paper, we propose an improved method for the design of FRM FIR filters with reduced passband group delays by directly imposing a group delay constraint. The proposed design method may be considered as a generalization of the method in [4], [5] by introducing an additional group delay constraint in the overall optimization. The FRM filter design problem is reformulated by taking into account both the complex magnitude error and the group delay error. The proposed approach is similar to but more complicated than our recent work on the design of FIR filters with reduced group delay error [6], [7]. One of the key steps of the design method proposed in this paper is the derivation of the group delay of an FRM filter and its gradient with respect to the subfilter coefficients. A group delay constraint is then incorporated into the overall optimization. Following [5], another key step is that the optimization is formulated as a second-order cone programming (SOCP) problem instead of a semidefinite programming (SDP) one [4], [6] in order to reduce the design time. The effectiveness and flexibility of the proposed design method will be illustrated by an example, and compared favorably with some of the existing design methods.

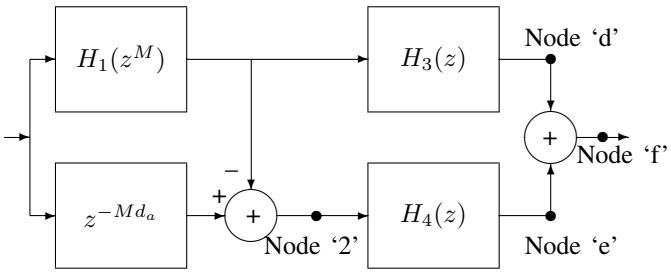

Fig. 1. An FRM filter structure.

II. Group Delay and Gradients for FRM Filters

Due to space limitation, in this paper we only discuss the design of a single stage FRM filter with a nonlinear phase prototype filter, a delay line and two linear phase masking filters, as illustrated in Fig. 1. The general case of multi-stage FRM filter with nonlinear phase masking filters will be reported elsewhere. Throughout this paper, we assume that the lengths of the subfilters are odd. Moreover, for simplicity, the frequency variable ω is omitted unless otherwise specified.

A. Group Delay for FRM Filters

Let $\mathbf{h}_1 = [h_1(0), \quad h_1(1), \quad \ldots, \quad h_1(N_1-1)]^T$ and $\mathbf{h}_l = \left[\ h_l(\frac{N_l-1}{2}), \quad 2h_l(\frac{N_l-1}{2}+1), \quad \ldots, \quad 2h_l(N_l-1)\ \right]^T$ $(l = 3, 4)$ denote the vectors of the coefficients associated with the prototype filter of length N_1 and the masking filters of lengths N_l $(l = 3, 4)$, respectively, where the masking filters are assumed to satisfy the symmetric condition: $h_l(n) = h_l(N_l - 1 - n)$ $(n = 0, 1, \ldots, \frac{N_l-1}{2})$. Let $A_i e^{j\phi_i}$ $(i = 1, 3, 4)$ be the frequency responses of $H_1(z^M)$, $H_3(z)$ and $H_4(z)$, respectively, where M is a scaling integer [1]. The magnitude

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and phase responses are

$$A_1 = \sqrt{(\mathbf{h}_1^T\mathbf{c}_1)^2 + (\mathbf{h}_1^T\mathbf{s}_1)^2}, \quad \phi_1 = \arctan 2(\mathbf{h}_1^T\mathbf{s}_1, \mathbf{h}_1^T\mathbf{c}_1),$$
$$A_l = \left|\mathbf{h}_l^T\mathbf{c}_l\right|, \quad \phi_l = -\left(\frac{N_m-1}{2}\right)\omega, \quad l = 3, 4,$$

where $N_m = \max\{N_3, N_4\}$, and $\mathbf{c}_1$, $\mathbf{s}_1$ and $\mathbf{c}_l$ ($l = 3, 4$) are trigonometric vectors

$$\mathbf{c}_1 = \left[\begin{array}{cccc} 1 & \cos M\omega & \ldots & \cos(N_1-1)M\omega \end{array}\right]^T, \tag{1a}$$
$$\mathbf{s}_1 = \left[\begin{array}{cccc} 0 & -\sin M\omega & \ldots & -\sin(N_1-1)M\omega \end{array}\right]^T, \tag{1b}$$
$$\mathbf{c}_l = \left[\begin{array}{cccc} 1 & \cos\omega & \cos 2\omega & \ldots \quad \cos\frac{N_l-1}{2}\,\omega \end{array}\right]^T, \quad l = 3, 4, \tag{1c}$$

and

$$\arctan 2(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & \text{for } x \geq 0, \\ \arctan\left(\frac{y}{x}\right) \pm \pi & \text{otherwise.} \end{cases}$$

The frequency responses at nodes '2', 'd', 'e' and 'f' in Fig. 1 can be obtained as

$$A_2 e^{j\phi_2} = e^{-jMd_a\omega} - A_1 e^{j\phi_1}, \quad A_d e^{j\phi_d} = A_1 A_3 e^{j(\phi_1+\phi_3)},$$
$$A_e e^{j\phi_e} = A_2 A_4 e^{j(\phi_2+\phi_4)}, \quad A_f e^{j\phi_f} = A_d e^{j\phi_d} + A_e e^{j\phi_e},$$

where d_a is the group delay of the prototype filter. To effectively reduce the group delay of the FRM filter, d_a should be less than $\frac{N_1-1}{2}$. Let $\mathcal{N}_f = A_d \sin\phi_d + A_e \sin\phi_e$ and $\mathcal{D}_f = A_d \cos\phi_d + A_e \cos\phi_e$. The phase response of the FRM filter is $\phi_f = \arctan 2(\mathcal{N}_f, \mathcal{D}_f)$. Using the derivative identity $\frac{d}{d\omega}\arctan 2(y, x) = \frac{d}{d\omega}\arctan(y/x) = \frac{1}{1+(y/x)^2}\frac{d(y/x)}{d\omega}$, the group delay of the FRM filter is given by

$$\tau_f(\omega) = -\frac{1}{\mathcal{D}_f^2 + \mathcal{N}_f^2}\left(\frac{d\mathcal{N}_f}{d\omega}\mathcal{D}_f - \mathcal{N}_f\frac{d\mathcal{D}_f}{d\omega}\right). \tag{2}$$

$\frac{d\mathcal{N}_f}{d\omega}$, $\frac{d\mathcal{D}_f}{d\omega}$ can be subsequently expressed as functions of $\frac{dA_i}{d\omega}$, $\frac{d\phi_i}{d\omega}$ ($i = 1, 3, 4$)

$$\frac{d\mathcal{N}_f}{d\omega} = \frac{dA_d}{d\omega}\sin\phi_d + A_d\cos\phi_d\frac{d\phi_d}{d\omega} + \frac{dA_e}{d\omega}\sin\phi_e + A_e\cos\phi_e\frac{d\phi_e}{d\omega}, \tag{3a}$$
$$\frac{d\mathcal{D}_f}{d\omega} = \frac{dA_d}{d\omega}\cos\phi_d - A_d\sin\phi_d\frac{d\phi_d}{d\omega} + \frac{dA_e}{d\omega}\cos\phi_e - A_e\sin\phi_e\frac{d\phi_e}{d\omega}, \tag{3b}$$
$$\frac{dA_d}{d\omega} = \frac{dA_1}{d\omega}A_3 + A_1\frac{dA_3}{d\omega}, \quad \frac{d\phi_d}{d\omega} = \frac{d\phi_1}{d\omega} + \frac{d\phi_3}{d\omega}, \tag{3c}$$
$$\frac{dA_e}{d\omega} = \frac{dA_2}{d\omega}A_4 + A_2\frac{dA_4}{d\omega}, \quad \frac{d\phi_e}{d\omega} = \frac{d\phi_2}{d\omega} + \frac{d\phi_4}{d\omega}. \tag{3d}$$

Denote $\tilde{\mathbf{s}}_1 = \frac{d\mathbf{s}_1}{d\omega}$ and $\tilde{\mathbf{c}}_i = \frac{d\mathbf{c}_i}{d\omega}$ ($i = 1, 3, 4$). It can be shown that

$$\frac{dA_1}{d\omega} = \frac{1}{A_1}\mathbf{h}_1^T\mathbf{P}_{m\omega 1}\mathbf{h}_1, \qquad \frac{d\phi_1}{d\omega} = \frac{1}{A_1^2}\mathbf{h}_1^T\mathbf{P}_{p\omega 1}\mathbf{h}_1, \tag{4a}$$
$$\frac{dA_2}{d\omega} = \frac{1}{A_2}\mathbf{h}_2^T\mathbf{P}_{m\omega 1}\mathbf{h}_2, \qquad \frac{d\phi_2}{d\omega} = \frac{1}{A_2^2}\mathbf{h}_2^T\mathbf{P}_{p\omega 1}\mathbf{h}_2, \tag{4b}$$
$$\frac{dA_l}{d\omega} = \mathrm{Sign}\left(\mathbf{h}_l^T\mathbf{c}_l\right)\,\mathbf{h}_l^T\tilde{\mathbf{c}}_l, \quad \frac{d\phi_l}{d\omega} = -\frac{N_m-1}{2}, \quad l = 3, 4, \tag{4c}$$

where $\mathbf{h}_2 = \mathbf{u}_{d_a+1} - \mathbf{h}_1$ ($\mathbf{u}_{d_a+1}$ is an all-zero vector except having 1 at the (d_a+1)-th entry), $\mathrm{Sign}(x)$ means the sign of x, and $\mathbf{P}_{m\omega 1}$ and $\mathbf{P}_{p\omega 1}$ are symmetric matrices obtained by

$$\mathbf{P}_{m\omega 1} = \frac{1}{2}\left(\tilde{\mathbf{P}}_{m\omega 1} + \tilde{\mathbf{P}}_{m\omega 1}^T\right), \quad \mathbf{P}_{p\omega 1} = \frac{1}{2}\left(\tilde{\mathbf{P}}_{p\omega 1} + \tilde{\mathbf{P}}_{p\omega 1}^T\right), \tag{5a}$$
$$\tilde{\mathbf{P}}_{m\omega 1} = \mathbf{c}_1\tilde{\mathbf{c}}_1^T + \mathbf{s}_1\tilde{\mathbf{s}}_1^T, \quad \tilde{\mathbf{P}}_{p\omega 1} = \tilde{\mathbf{s}}_1\mathbf{c}_1^T - \tilde{\mathbf{c}}_1\mathbf{s}_1^T. \tag{5b}$$

The analytic expression of the group delay of an FRM filter given in (2)-(5) will facilitate the derivation of gradient of the group delay with respect to the subfilter coefficients.

B. Gradients for FRM Filters

For an FRM filter with reduced group delay, both the frequency response and the group delay are nonlinear with respect to the subfilter coefficients.

Gradient of the Frequency Response

The frequency response of an FRM filter is

$$\begin{aligned} H_f(\omega) &= H_1(M\omega)H_3(\omega) + \left(e^{-jMd_a\omega} - H_1(M\omega)\right)H_4(\omega) \\ &= e^{-jd_m\omega}\mathbf{h}_1^T\mathbf{e}_1(\mathbf{h}_3^T\mathbf{c}_3 - \mathbf{h}_4^T\mathbf{c}_4) \\ &\quad + e^{-j(Md_a+d_m)\omega}\mathbf{h}_4^T\mathbf{c}_4, \end{aligned} \tag{6}$$

where $\mathbf{e}_1 = \left[\begin{array}{cccc} 1 & e^{-jM\omega} & e^{-j2M\omega} & \ldots \quad e^{-j(N_1-1)M\omega} \end{array}\right]^T$ and $d_m = \frac{N_m-1}{2}$. When the design variables are put together as a design vector $\mathbf{x} = [\mathbf{h}_1 \;\; \mathbf{h}_3 \;\; \mathbf{h}_4]$ of dimension N_T where $N_T = N_1 + \frac{N_3+1}{2} + \frac{N_4+1}{2}$, the gradient of the frequency response with respect to $\mathbf{x}$ is given by

$$\mathbf{g}_m(\omega) = \begin{bmatrix} \frac{dH_f(\omega)}{d\mathbf{h}_1} \\ \frac{dH_f(\omega)}{d\mathbf{h}_3} \\ \frac{dH_f(\omega)}{d\mathbf{h}_4} \end{bmatrix} = \begin{bmatrix} e^{-jd_m\omega}(\mathbf{h}_3^T\mathbf{c}_3 - \mathbf{h}_4^T\mathbf{c}_4)\mathbf{e}_1 \\ e^{-jd_m\omega}\mathbf{h}_1^T\mathbf{e}_1\mathbf{c}_3 \\ e^{-jd_m\omega}\left(e^{-jMd_a\omega} - \mathbf{h}_1^T\mathbf{e}_1\right)\mathbf{c}_4 \end{bmatrix}. \tag{7}$$

Gradient of the Group Delay

Using the group delay expressions given in (2)-(5), the gradient of the group delay with respect to $\mathbf{x}$ is defined as

$$\mathbf{g}_\tau(\omega) = \left[\left(\frac{d\tau_f(\omega)}{d\mathbf{h}_1}\right)^T \quad \left(\frac{d\tau_f(\omega)}{d\mathbf{h}_3}\right)^T \quad \left(\frac{d\tau_f(\omega)}{d\mathbf{h}_4}\right)^T\right]^T, \tag{8}$$

whose components, $\frac{d\tau_f(\omega)}{d\mathbf{h}_i}$ ($i = 1, 3, 4$) are given by

$$\begin{aligned} \frac{d\tau_f(\omega)}{d\mathbf{h}_i} &= \frac{2}{(\mathcal{D}_f^2 + \mathcal{N}_f^2)^2}\left(\mathcal{D}_f\frac{d\mathcal{D}_f}{d\mathbf{h}_i} + \mathcal{N}_f\frac{d\mathcal{N}_f}{d\mathbf{h}_i}\right) \\ &\quad \cdot\left(\frac{d\mathcal{N}_f}{d\omega}\mathcal{D}_f - \mathcal{N}_f\frac{d\mathcal{D}_f}{d\omega}\right) - \frac{1}{\mathcal{D}_f^2 + \mathcal{N}_f^2}\left[\frac{d^2\mathcal{N}_f}{d\omega d\mathbf{h}_i}\mathcal{D}_f\right. \\ &\quad \left. + \frac{d\mathcal{N}_f}{d\omega}\frac{d\mathcal{D}_f}{d\mathbf{h}_i} - \frac{d^2\mathcal{D}_f}{d\omega d\mathbf{h}_i} - \frac{d\mathcal{D}_f}{d\omega}\frac{d\mathcal{N}_f}{d\mathbf{h}_i}\right], \quad i = 1, 3, 4, \end{aligned} \tag{9}$$

where $\frac{d\mathcal{N}_f}{d\mathbf{h}_i}$, $\frac{d\mathcal{D}_f}{d\mathbf{h}_i}$ $\frac{d^2\mathcal{N}_f}{d\omega d\mathbf{h}_i}$ and $\frac{d^2\mathcal{D}_f}{d\omega d\mathbf{h}_i}$ ($i = 1, 3, 4$) can be expressed as functions of $\frac{dA_i}{d\mathbf{h}_i}$, $\frac{d\phi_i}{d\mathbf{h}_i}$, $\frac{d^2A_i}{d\omega d\mathbf{h}_i}$ and $\frac{d^2\phi_i}{d\omega d\mathbf{h}_i}$ ($i = 1, 3, 4$). Due to space limitation, only the expressions for $i = 3, 4$ are given as following. The expressions for $i = 1$ are

similar but more lengthy and omitted here. They are available at http://www.ntu.edu.sg/home/ezplin/ISCAS07.htm.

$$\frac{d\mathcal{N}_f}{d\mathbf{h}_i} = A_l \frac{dA_i}{d\mathbf{h}_i}\sin\phi_q, \quad \frac{d\mathcal{D}_f}{d\mathbf{h}_i} = A_l \frac{dA_i}{d\mathbf{h}_i}\cos\phi_q, \tag{10a}$$

$$\frac{d^2\mathcal{N}_f}{d\omega d\mathbf{h}_i} = \left(\frac{dA_l}{d\omega}\frac{dA_i}{d\mathbf{h}_i} + A_l\frac{d^2A_i}{d\omega d\mathbf{h}_i}\right)\sin\phi_q + A_l\frac{dA_i}{d\mathbf{h}_i}\cos\phi_q\left(\frac{d\phi_l}{d\omega} + \frac{d\phi_i}{d\omega}\right), \tag{10b}$$

$$\frac{d^2\mathcal{D}_f}{d\omega d\mathbf{h}_i} = \left(\frac{dA_l}{d\omega}\frac{dA_i}{d\mathbf{h}_i} + A_l\frac{d^2A_i}{d\omega d\mathbf{h}_i}\right)\cos\phi_q - A_l\frac{dA_i}{d\mathbf{h}_i}\sin\phi_q\left(\frac{d\phi_l}{d\omega} + \frac{d\phi_i}{d\omega}\right), \tag{10c}$$

for $(i,l,q) = (3,1,d)$ or $(i,l,q) = (4,2,e)$.

$\frac{dA_i}{d\mathbf{h}_i}$, $\frac{d\phi_i}{d\mathbf{h}_i}$, $\frac{d^2A_i}{d\omega d\mathbf{h}_i}$ and $\frac{d^2\phi_i}{d\omega d\mathbf{h}_i}$ $(i = 1,3,4)$ are given by

$$\frac{dA_1}{d\mathbf{h}_1} = \frac{1}{A_1}\mathbf{P}_{mh1}\mathbf{h}_1, \quad \frac{d^2A_1}{d\omega d\mathbf{h}_1} = \frac{-1}{A_1}\frac{dA_1}{d\mathbf{h}_1}\frac{dA_1}{d\omega} + \frac{2}{A_1}\mathbf{P}_{m\omega 1}\mathbf{h}_1, \tag{11a}$$

$$\frac{d\phi_1}{d\mathbf{h}_1} = \frac{1}{A_1^2}\mathbf{P}_{ph1}\mathbf{h}_1, \quad \frac{d^2\phi_1}{d\omega d\mathbf{h}_1} = \frac{-2}{A_1}\frac{dA_1}{d\mathbf{h}_1}\frac{d\phi_1}{d\omega} + \frac{2}{A_1^2}\mathbf{P}_{p\omega 1}\mathbf{h}_1, \tag{11b}$$

$$\frac{dA_l}{d\mathbf{h}_l} = \mathrm{Sign}(\mathbf{h}_l^T\mathbf{c}_l)\mathbf{c}_l, \quad \frac{d^2A_l}{d\omega d\mathbf{h}_l} = \mathrm{Sign}(\mathbf{h}_l^T\mathbf{c}_l)\tilde{\mathbf{c}}_l, \quad l = 3,4, \tag{11c}$$

where $\mathbf{P}_{mh1} = \mathbf{c}_1\mathbf{c}_1^T + \mathbf{s}_1\mathbf{s}_1^T$ and $\mathbf{P}_{ph1} = \mathbf{s}_1\mathbf{c}_1^T - \mathbf{c}_1\mathbf{s}_1^T$.

III. Overall Optimization for FRM Filters

The design of an FRM filter involves two steps. First, a suboptimal FRM filter is designed with a group of subfilters [1]. Second, overall optimization is carried out to optimize the performance of the FRM filter [4], [5], [8].

Let $H_{des}(\omega)$, $\tau_{des}(\omega)$ be the desired frequency response and passband group delay, and $H_f(\omega,\mathbf{x})$ and $\tau_f(\omega,\mathbf{x})$ be the designed frequency response and passband group delay that depend on the design vector $\mathbf{x}$.

Problem Formulation

Denote the complex magnitude error, $\eta_m(\omega,\mathbf{x}) = |H_f(\omega,\mathbf{x}) - H_{des}(\omega)|$ for $\omega \in \Omega_m$ and the group delay error, $\eta_g(\omega,\mathbf{x}) = |\tau_f(\omega,\mathbf{x}) - \tau_{des}(\omega)|$ for $\omega \in \Omega_g$, where Ω_m and Ω_g are frequency bands of interest for the magnitude response and the group delay, respectively. The design of an FRM filter with reduced group delay error can be formulated as

$$\underset{\mathbf{x}}{\text{minimize}}\left\{\text{maximize}\left\{\max_{\omega\in\Omega_m}\eta_m(\omega,\mathbf{x}), \max_{\omega\in\Omega_g}\beta\eta_g(\omega,\mathbf{x})\right\}\right\}, \tag{12}$$

where β is a positive constant balancing the complex magnitude and group delay errors. Note that $\beta = 0$ reduces to the conventional problem of magnitude optimal FRM filter design [5].

Estimation Errors for FRM Filters

Suppose we have a reasonable initial filter $\mathbf{x}_0$ to start optimization, and we are in the k-th iteration. For a nonlinear and smooth $H_f(\omega,\mathbf{x})$ at the vicinity of $\mathbf{x}_k$, *i.e.,* at $\mathbf{x}_k + \mathbf{v}_k$, where $\mathbf{v}_k$ is a small vector, the complex magnitude error becomes [4], [5]

$$\begin{aligned}\eta_m(\omega,\mathbf{x}_k+\mathbf{v}_k) &\approx \left|\mathbf{g}_{m,k}^T(\omega)\mathbf{v}_k + H_f(\omega,\mathbf{x}_k) - H_{des}(\omega)\right| \\ &= \left|\left(\mathbf{g}_{mr,k}^T(\omega)\mathbf{v}_k + e_{mr,k}(\omega)\right) + j\left(\mathbf{g}_{mi,k}^T(\omega)\mathbf{v}_k + e_{mi,k}(\omega)\right)\right|\end{aligned} \tag{13}$$

where $\mathbf{g}_{m,k}(\omega) = \mathbf{g}_{mr,k} + j\mathbf{g}_{mi,k}$ is the gradient of $H_f(\omega,\mathbf{x})$ with respect to $\mathbf{x}$ in (7) and evaluated at $\mathbf{x}_k$, and $e_{mr,k}(\omega) = H_{rf}(\omega,\mathbf{x}_k) - H_{rd}(\omega)$, $e_{mi,k}(\omega) = H_{if}(\omega,\mathbf{x}_k) - H_{id}(\omega)$ with $H_{rf}(\omega,\mathbf{x}_k)$, $H_{rd}(\omega)$ and $H_{if}(\omega,\mathbf{x}_k)$, $H_{id}(\omega)$ being the real and imaginary parts of $H_f(\omega,\mathbf{x}_k)$ and $H_{des}(\omega)$, respectively.

Similarly, the group delay error at the k-th iteration can be obtained as

$$\begin{aligned}\eta_g(\omega,\mathbf{x}_k+\mathbf{v}_k) &\approx \left|\mathbf{g}_{\tau,k}^T(\omega)\mathbf{v}_k + \tau_f(\omega,\mathbf{x}_k) - \tau_{des}(\omega)\right| \\ &= \left|\mathbf{g}_{\tau,k}^T(\omega)\mathbf{v}_k + e_{\tau,k}(\omega)\right|,\end{aligned} \tag{14}$$

where $\mathbf{g}_{\tau,k}(\omega)$ is the gradient of the group delay with respect to $\mathbf{x}$ in (8) and evaluated at $\mathbf{x}_k$ and $e_{\tau,k}(\omega) = \tau_f(\omega,\mathbf{x}_k) - \tau_{des}(\omega)$.

SOCP Formulation

Using (13) and (14), it follows that an approximated solution of (12) at the k-th iteration can be obtained by solving the following problem

$$\text{minimize} \quad \eta \tag{15a}$$

subject to :

$$\|\mathbf{G}_{m,k}(\omega)\mathbf{v}_k + \mathbf{e}_{m,k}(\omega)\| \le \eta \quad \text{for} \quad \omega\in\Omega_m, \tag{15b}$$

$$\beta\left\|\left(\mathbf{g}_{\tau,k}^T(\omega)\mathbf{v}_k + e_{\tau,k}(\omega)\right)\right\| \le \eta \quad \text{for} \quad \omega\in\Omega_g, \tag{15c}$$

$$\|\mathbf{v}_k\| \le b_{\mathbf{v}}, \tag{15d}$$

where $\|x\|$ represents norm of x, $b_{\mathbf{v}}$ is a prescribed bound for $\mathbf{v}_k$, $\mathbf{e}_{m,k}(\omega) = [e_{mr,k}(\omega) \;\; e_{mi,k}(\omega)]^T$ and $\mathbf{G}_{m,k}(\omega) = [\mathbf{g}_{mr,k} \;\; \mathbf{g}_{mi,k}]^T$. Noted that (15c) is the group delay constraint.

Define $\mathbf{y}_k = [\eta \;\; \mathbf{v}_k^T]^T$ and $\mathbf{u}_1 = [1 \;\; 0 \;\; \ldots \;\; 0]^T$. The constrained optimization problem defined by (15) can then be formulated as an SOCP problem by digitizing the frequency variable ω over a dense set of frequencies in the bands of interest. With m_K discrete frequency grids $\{\omega_1, \ldots, \omega_{m_K}\} \subset \Omega_m$ and g_K discrete frequency grids $\{\omega_1, \ldots, \omega_{g_K}\} \subset \Omega_g$, the k-th iteration problem in (15) becomes

$$\text{minimize} \quad \mathbf{u}_1^T\mathbf{y}_k \tag{16a}$$

subject to :

$$\begin{bmatrix}1 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{G}_{m,k}(\omega_l)\end{bmatrix}\mathbf{y}_k + \begin{bmatrix}0 \\ \mathbf{e}_{m,k}(\omega_l)\end{bmatrix} \in \mathcal{C}_l, \quad l = 1,\ldots,m_K, \tag{16b}$$

$$\begin{bmatrix}1 & \mathbf{0}^T \\ 0 & \beta\mathbf{g}_{\tau,k}^T(\omega_i)\end{bmatrix}\mathbf{y}_k + \begin{bmatrix}0 \\ \beta e_{\tau,k}(\omega_i)\end{bmatrix} \in \mathcal{C}_i, \quad i = 1,\ldots,g_K, \tag{16c}$$

$$\begin{bmatrix}0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{I}\end{bmatrix}\mathbf{y}_k + \begin{bmatrix}b_{\mathbf{v}} \\ \mathbf{0}\end{bmatrix} \in \mathcal{C}_p, \tag{16d}$$

where $\mathbf{I}$ is an identity matrix of size N_T, and $\mathcal{C}_l$, $\mathcal{C}_i$ and $\mathcal{C}_p$ are the second-order cones of dimensions 3, 2 and $N_T + 1$,

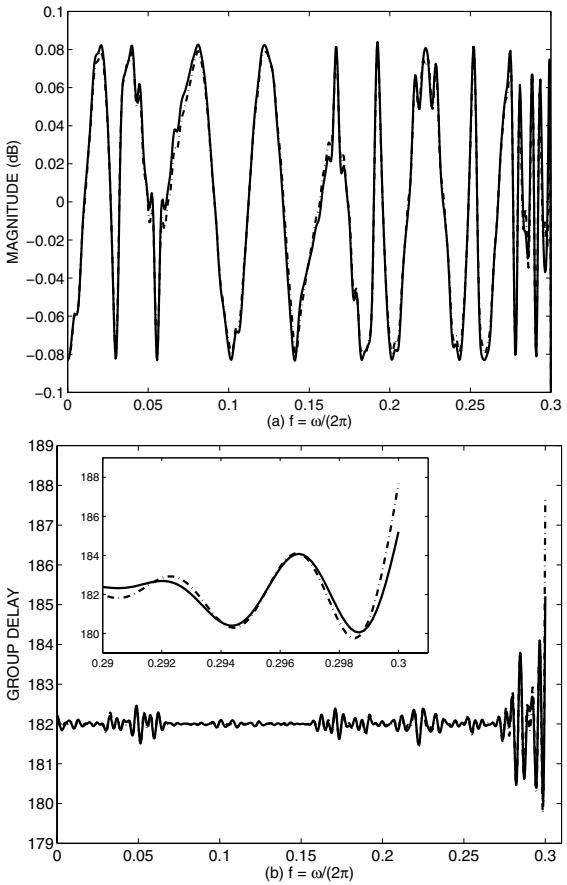

Fig. 2. (a) Passband ripple and (b) passband group delay of the FRM filters. The proposed FRM filter ($\beta = 3 \times 10^{-3}$) is indicated in solid lines while the magnitude optimal FRM filter is in broken lines.

respectively. This problem can be solved efficiently by any SOCP solvers, such as SeDuMi [9] used in the simulations.

Having solved the problem in (16) for a minimizer $\mathbf{y}_k^* = [\eta_k^* \quad \mathbf{v}_k^{*T}]^T$, vector $\mathbf{v}_k^*$ is used to update $\mathbf{x}_k$ as $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{v}_k^*$. The iteration continues until variation of η becomes insignificantly small.

IV. Design Example

An FRM filter with cutoff frequencies $\omega_p = 0.6\pi$, $\omega_s = 0.61\pi$ [1] and desired passband group delay $\tau_{des}(\omega) = 182$ is considered here. To compare the performance of the designed filters, some important parameters, such as passband ripple, δ_p, stopband ripple, δ_s, both in decibels, and relative deviation in passband group delay, defined as $D_{rd} = \max_{\omega \in \Omega_g} \left| \frac{\tau_f(\omega) - \tau_{des}(\omega)}{\tau_{des}(\omega)} \right|$, are adopted.

Given $M = 9$, a suboptimal FRM filter with passband group delay around 182 and $D_{rd} = 0.0332$ is designed using the classical FRM design method [1] with $N_1 = 45$, $N_3 = 41$, $N_4 = 33$, $d_a = 18$ and $d_m = 20$. Using this suboptimal filter as an initial design, a magnitude optimal FRM filter is obtained with $D_{rd} = 0.0313$ using the method in [5] or equivalently the proposed method with $\beta = 0$. For the overall optimization, 900 frequency grids are selected and relatively dense grid points are placed in the regions near the bandedges. Using this magnitude optimal FRM filter as an initial design, two optimal FRM filters are designed using the proposed method with $\beta = 2 \times 10^{-3}$ and 3×10^{-3}, respectively, and $b_{\mathbf{v}} = 1 \times 10^{-4} N_T$ ($N_T = 83$).

As it can be seen from Table I, compared with the magnitude optimal FRM filter, for $\beta = 2 \times 10^{-3}$, the relative deviation in passband group delay of the proposed FRM filter is reduced by 0.006, which corresponds to 18% reduction. However, the passband and stopband ripples are about 0.001 dB and 0.07 dB larger. For $\beta = 3 \times 10^{-3}$, D_{rd} is reduced by 0.014, which corresponds to 43% reduction. However, the passband and stopband ripples increase by 0.04 dB and 0.4 dB, respectively. The performance of the proposed FRM filter for $\beta = 3 \times 10^{-3}$ is also compared with that of the magnitude optimal FRM filter in Fig. 2, which shows that significant reduction in group delay error is achieved at the cost of slight increase in magnitude error.

The above example has clearly demonstrated the flexibility and effectiveness of the proposed design method in controlling the group delay error by adjusting β, which balances the complex magnitude and the group delay errors.

TABLE I
Design results of the FRM filters.

Filter	β	D_{rd}	δ_p (dB)	δ_s (dB)
Suboptimal filter [1]	(NA)	0.0332	0.0845	-40.2190
Magnitude optimal filter [5]	0	0.0313	0.0804	-40.6742
Proposed filter	2×10^{-3}	0.0255	0.0811	-40.6049
Proposed filter	3×10^{-3}	0.0176	0.0839	-40.2662

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