Synthesis of Very Sharp Hilbert Transformer Using the Frequency-Response Masking Technique

Y. C. Lim and Y. J. Yu

***Abstract*—Frequency translation of audio signals requires the use of very sharp Hilbert transformer if the low frequency components are to be faithfully translated. Unfortunately, the order of a very sharp Hilbert transformer is very high, resulting in very high arithmetic complexity. The frequency-response masking (FRM) technique has been used very successfully for greatly reducing the complexity of extremely sharp halfband filters. There is a very close relationship between a halfband filter and a Hilbert transformer. In this correspondence, we propose to synthesize a very sharp Hilbert transformer using the FRM technique via the synthesis of a very sharp halfband filter.**

***Index Terms*—FIR digital filter, Hilbert transformer, frequency-response masking.**

I. Introduction

The frequency-response masking (FRM) technique [1]–[23] has been used very successfully for the synthesis of very sharp filters with very high computational efficiency. In order to simplify notation, we will ignore causality (which can be restored easily by introducing appropriate delays) by assuming that all filters are zero-phase filters. In the classical FRM technique [1], the z-transform transfer function $H(z)$ of a filter is synthesized as a system of subfilters as in

$$H(z) = H_{Ma}(z)H_a(z^M) + (1 - H_a(z^M))H_{Mc}(z) \tag{1}$$

where $H_a(z^M)$ is the z-transform transfer function of a filter derived by replacing each delay of a prototype band edge shaping filter by M delays. The functions $H_{Ma}(z)$ and $H_{Mc}(z)$ are the z-transform transfer functions of the masking filters. The synthesis structure for (1) is shown in Fig. 1.

II. Halfband Filter Synthesized Using the FRM Technique

In [4], a method for the synthesis of halfband filter using the frequency-response masking technique was proposed. For the purpose of defining the notations used in this correspondence, the essential points in [4] are summarized as follows. Let $H_a(z)$ be the z-transform transfer function of a prototype unity-gain halfband band edge shaping filter given by

$$H_a(z) = \frac{1}{2} + A(z). \tag{2}$$

Let the length of $H_{Ma}(z)$ be $2K + 1$. Thus, $H_{Ma}(z)$ may be written as

$$H_{Ma}(z) = h_{Ma}(0) + \sum_{k=1}^{K} h_{Ma}(k)\left(z^k + z^{-k}\right). \tag{3}$$

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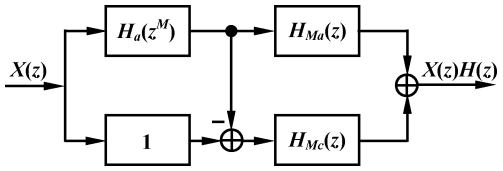

Fig. 1. Structure of a filter synthesized using the FRM technique. The transfer function "1" may be replaced by $z^{-\bullet}$ to restore causality.

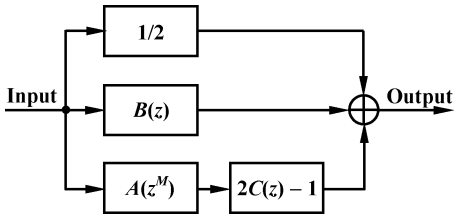

Fig. 2. Structure for synthesizing a halfband filter using the FRM technique.

Define the z-transform transfer function $B(z)$ and $C(z)$ as in (4) and (5), respectively.

$$B(z) = h_{Ma}(1)(z + z^{-1}) + h_{Ma}(3)(z^3 + z^{-3}) + \cdots \tag{4}$$

$$C(z) = h_{Ma}(0) + h_{Ma}(2)(z^2 + z^{-2}) + h_{Ma}(4)(z^4 + z^{-4}) + \cdots. \tag{5}$$

Substituting (4) and (5) into (3), we have

$$H_{Ma}(z) = B(z) + C(z). \tag{6}$$

In [4], $H_{Mc}(z)$ is given by

$$H_{Mc}(z) = 1 - H_{Ma}(-z). \tag{7}$$

From (6) and (7), and noting that $B(-z) = -B(z)$ and that $C(-z) = C(z)$, we have

$$H_{Mc}(z) = 1 + B(z) - C(z). \tag{8}$$

Substituting (2), (6), and (8) into (1), we have

$$H(z) = \frac{1}{2} + B(z) + A(z^M)(2C(z) - 1). \tag{9}$$

The halfband transfer function $H(z)$ synthesized using (9) is shown in Fig. 2, where M is odd.

III. Deriving the Hilbert Transformer From the Halfband Filter

A Hilbert transformer may be derived from a unity-gain halfband filter by subtracting the constant ½ from its transfer function and then modulating the remaining coefficients by $e^{jn\pi/2}$; the effect of this action on the frequency response is shown in Fig. 3. This will produce an odd-length Hilbert transformer with a passband gain of 0.5. For an odd length Hilbert transformer, every other one of its impulse responses is zero; an even length Hilbert transformer may be derived by dropping these zero coefficients.

The transfer function of the Hilbert transformer $H_H(z)$ is given by

$$H_H(z) = 2B(jz) + 2A(j^M z^M)(2C(jz) - 1). \tag{10}$$

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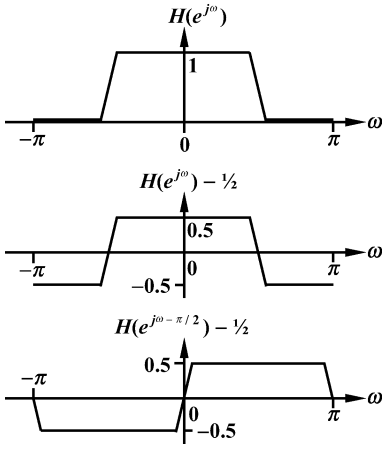

Fig. 3. Frequency response of a Hilbert transformer derived from a halfband filter.

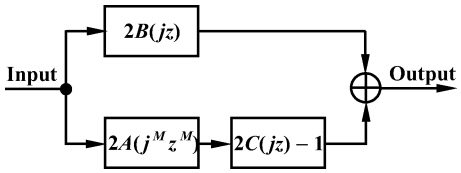

Fig. 4. Structure of a Hilbert transformer implemented using the FRM technique.

The implementation structure is shown in Fig. 4.

In (10), the transfer functions $A(j^M z^M)$, $B(jz)$, and $2C((jz)-1)$ may be written as

$$A(j^M z^M) = j \times A_0(z) \tag{11a}$$

$$B(jz) = j \times B_0(z) \tag{11b}$$

$$2C(jz) - 1 = j \times C_0(z) \tag{11c}$$

where $A_0(z)$, $B_0(z)$, and $C_0(z)$ are zero phase z-transform transfer functions. Thus, $H_H(z)$ may be written as

$$H_H(z) = j \times H_{H0}(z) \tag{12}$$

where $H_{H0}(z)$ is a zero phase z-transform transfer function. The factor j in (12) represents the 90° phase shift of the Hilbert transformer. The factor j arises because a zero phase $H(z)$ [see (1)] is used in the derivation. The factor j does not exist if a causal halfband filter is used as the prototype filter. Consider using the causal halfband filter with linear phase z-transform transfer function $z^{-\text{odd}}H(z)$ as the prototype filter. After replacing z with jz, the z-transform transfer function of the Hilbert transformer will become

$$(jz)^{-\text{odd}}H_H(z) = j \times z^{-\text{odd}}(2B(jz) + 2A(j^M z^M)(2C(jz) - 1))$$

and will not have the factor j; the required 90° phase shift is inherent in the transfer function.

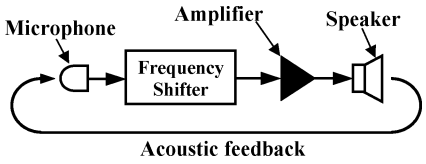

Fig. 5. Frequency shifter is inserted to suppress oscillation due to acoustic feedback.

TABLE I
COEFFICIENT VALUES OF $H_a(z)$

$h_a(-57) = 0.00000885 = h_a(57)$	$h_a(-27) = -0.00056200 = h_a(27)$
$h_a(-55) = -0.00000449 = h_a(55)$	$h_a(-25) = 0.00068783 = h_a(25)$
$h_a(-53) = 0.00001756 = h_a(53)$	$h_a(-23) = -0.00083008 = h_a(23)$
$h_a(-51) = -0.00001766 = h_a(51)$	$h_a(-21) = 0.00100667 = h_a(21)$
$h_a(-49) = 0.00003458 = h_a(49)$	$h_a(-19) = -0.00121242 = h_a(19)$
$h_a(-47) = -0.00004162 = h_a(47)$	$h_a(-17) = 0.00146971 = h_a(17)$
$h_a(-45) = 0.00006479 = h_a(45)$	$h_a(-15) = -0.00178378 = h_a(15)$
$h_a(-43) = -0.00008143 = h_a(43)$	$h_a(-13) = 0.00219478 = h_a(13)$
$h_a(-41) = 0.00011413 = h_a(41)$	$h_a(-11) = -0.00275833 = h_a(11)$
$h_a(-39) = -0.00014371 = h_a(39)$	$h_a(-9) = 0.00369024 = h_a(9)$
$h_a(-37) = 0.00019008 = h_a(37)$	$h_a(-7) = -0.05641928 = h_a(7)$
$h_a(-35) = -0.00023681 = h_a(35)$	$h_a(-5) = 0.01421025 = h_a(5)$
$h_a(-33) = 0.00030199 = h_a(33)$	$h_a(-3) = -0.05641928 = h_a(3)$
$h_a(-31) = -0.00037143 = h_a(31)$	$h_a(-1) = 0.29595677 = h_a(1)$
$h_a(-29) = 0.00046212 = h_a(29)$	$h_a(0) = 0.50000000$

TABLE II
COEFFICIENT VALUES OF $H_{Ma}(z)$

$h_{Ma}(-34) = -0.00695946 = h_{Ma}(34)$	$h_{Ma}(-16) = -0.09554157 = h_{Ma}(16)$
$h_{Ma}(-33) = -0.00001780 = h_{Ma}(33)$	$h_{Ma}(-15) = 0.29831766 = h_{Ma}(15)$
$h_{Ma}(-32) = 0.01728009 = h_{Ma}(32)$	$h_{Ma}(-14) = 0.11868809 = h_{Ma}(14)$
$h_{Ma}(-31) = 0.00012423 = h_{Ma}(31)$	$h_{Ma}(-13) = -0.27023419 = h_{Ma}(13)$
$h_{Ma}(-30) = -0.02805677 = h_{Ma}(30)$	$h_{Ma}(-12) = -0.23909232 = h_{Ma}(12)$
$h_{Ma}(-29) = -0.00018857 = h_{Ma}(29)$	$h_{Ma}(-11) = 0.23203805 = h_{Ma}(11)$
$h_{Ma}(-28) = 0.03361170 = h_{Ma}(28)$	$h_{Ma}(-10) = 0.30992696 = h_{Ma}(10)$
$h_{Ma}(-27) = 0.02809509 = h_{Ma}(27)$	$h_{Ma}(-9) = -0.18133683 = h_{Ma}(9)$
$h_{Ma}(-26) = -0.05102311 = h_{Ma}(26)$	$h_{Ma}(-8) = -0.39206726 = h_{Ma}(8)$
$h_{Ma}(-25) = -0.06916630 = h_{Ma}(25)$	$h_{Ma}(-7) = 0.19651044 = h_M(7)$
$h_{Ma}(-24) = 0.06194209 = h_{Ma}(24)$	$h_{Ma}(-6) = 0.39309235 = h_{Ma}(6)$
$h_{Ma}(-23) = 0.11227940 = h_{Ma}(23)$	$h_{Ma}(-5) = -0.05509426 = h_{Ma}(5)$
$h_{Ma}(-22) = -0.07434883 = h_{Ma}(22)$	$h_{Ma}(-4) = -0.41030147 = h_{Ma}(4)$
$h_{Ma}(-21) = -0.13452774 = h_{Ma}(21)$	$h_{Ma}(-3) = -0.03690918 = h_{Ma}(3)$
$h_{Ma}(-20) = 0.03216557 = h_{Ma}(20)$	$h_{Ma}(-2) = 0.41307057 = h_{Ma}(2)$
$h_{Ma}(-19) = 0.20409112 = h_{Ma}(19)$	$h_{Ma}(-1) = 0.31313419 = h_{Ma}(1)$
$h_{Ma}(-18) = 0.02892611 = h_{Ma}(18)$	$h_{Ma}(0) = 0.00000000$
$h_{Ma}(-17) = -0.24842371 = h_{Ma}(17)$	

IV. EXAMPLE

A problem often encountered in an acoustic feedback system is oscillation due to feedback. Such oscillation can be suppressed by inserting a frequency shifter into the system, as shown in Fig. 5. The frequency shifter shifts the frequency by an adjustable amount ranging from 0 to 5 Hz. A shift of 2 Hz produces no noticeable distortion to most people. The implementation of the shifter requires a sharp Hilbert transformer if low-frequency components must also be faithfully shifted.

In a particular system, the application required a Hilbert transformer with the following specifications: sampling rate = 32 kHz,

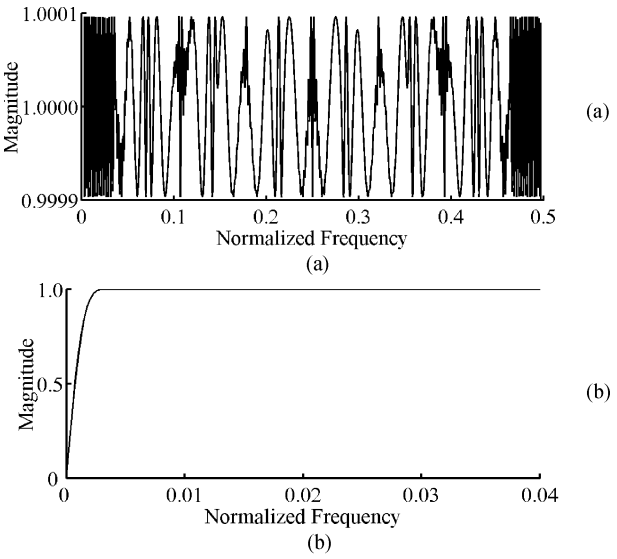

Fig. 6. Frequency response plots for the Hilbert transformer derived from the halfband filter whose impulse responses are shown in Tables I and II. (a) Passband response plot. (b) Frequency response expanded to show the transition band.

lower band edge $= 100$ Hz, peak ripple magnitude $= 0.0001$. These requirements correspond to a Hilbert transformer with transition width $0.003\,125 f_s$, where f_s is the sampling frequency. The estimated length of a direct form Chebyshev optimum design meeting the specification was 803.

Using the FRM technique, the prototype halfband filter has transition width $0.006\,25 f_s$. The coefficient values for a design with $M = 7$ are shown in Tables I and II. The frequency response of the resulting Hilbert transformer is shown in Fig. 6. The FRM technique reduces the number of multipliers by a factor of 3 when compared with the Chebyshev optimum design.

V. Conclusions

A new method for the synthesis of very sharp Hilbert transformer using the FRM technique is presented. The FRM technique produces significant reduction in hardware complexity when the transition width is narrow.

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