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TWO CLASSES OF FREQUENCY-RESPONSE MASKING LINEAR-PHASE FIR FILTERS FOR INTERPOLATION AND DECIMATION*

Håkan Johansson[1]

Abstract. This paper introduces two classes of frequency-response masking (FRM) linear-phase finite (length) impulse response (FIR) filters for interpolation and decimation by arbitrary integer factors M. As they are based on the FRM approach, the proposed filters are low-complexity (efficient) sharp-transition linear-phase FIR interpolation and decimation filters. Compared to previously existing FRM linear-phase FIR filter classes for interpolation and decimation, the new ones offer lower complexity and more freedom in selecting the locations of the passband and stopband edges. Furthermore, the proposed classes of FRM filters can, as special cases, realize efficient Mth-band FRM linear-phase FIR interpolation and decimation filters for all values of M. Previously, only half-band ($M = 2$) FRM linear-phase FIR filters have appeared in the literature. The paper includes design techniques suitable for the new filters and design examples illustrating their efficiency.
Key words: Linear-phase FIR filters, Mth-band filters, frequency-response masking, interpolation, decimation, optimization.

1. Introduction

The frequency-response masking (FRM) approach is a technique for realizing sharp-transition filters with low complexity [2], [5], [7], [12], [13], [16], [22], [23], [25]–[28], [31], [34], [35], [38], [42], [43].[2] The overall sharp-transition FRM filter is obtained by making use of periodic [3] model filters and nonperiodic masking filters with transition bands wider than that of the overall filter. This results in low-order subfilters and thereby low-complexity subfilters and in the end a low-complexity overall filter. Low complexity leads to low-power filter implementations, which are vital in battery-powered equipment like hand-held

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[1] Department of Electrical Engineering, Linköping University, Sweden. E-mail: hakanj@isy.liu.se

[2] The term low-complexity (efficient) filters means that the number of arithmetic operations (multiplications and additions) per sample required to implement the filter structure is small.

[3] A digital filter is said to be periodic if its frequency response is periodic with period $2\pi/L$, where L is a positive integer greater than one.

wireless communication systems. The FRM approach is particularly efficient for realizing FIR filters with sharp transition bands because the order, and thereby the complexity, of direct-form FIR realizations is proportional to the reciprocal of the transition bandwidth [10], [11], [19].

Since its introduction in [22], where linear-phase FIR filters with no additional restrictions were considered, the FRM approach has been successfully extended to many other filter categories. The FIR filters include low-delay filters [2], [42], approximately power-complementary filter pairs [13], half-band filters [38], Hilbert transformers [27], interpolation and decimation filters [26], two-channel filter banks (FBs) [16], [34], modulated multichannel FBs [7], [31], [35], [43], modulated transmultiplexers [5], and two-dimensional filters [23], [25]. The FRM approach has also been extended to infinite impulse response (IIR) filters as a means to obtain efficient high-speed recursive filters [18], approximately linear-phase filters [18], [28], interpolators and decimators for sampling rate conversion by factors of two [17], and Hilbert transformers [12]. However, there are still contexts that have not yet fully utilized the FRM approach. That is, by extending the FRM approach further, it is possible to reduce the complexity even more compared to existing solutions, and to do this for several different signal processing functions. This paper targets interpolation and decimation, which are important functions required in many different contexts [3], [6], [44].

The FRM approach is general in the sense that a filter can be said to be an FRM filter whenever its transfer function takes on a certain form that makes use of periodic model filters and nonperiodic masking filters (see Section 2). However, different choices of model and masking filters lead to FRM filters that have very different properties and are suitable in different contexts. This paper introduces two classes of FRM linear-phase FIR filters suitable for interpolation and decimation. The paper considers lowpass filters only, but the extensions to highpass and bandpass filters are relatively straightforward. Lowpass filters aimed for interpolation/decimation have a bandwidth of approximately π/M. Throughout this paper, such filters, with no additional restrictions, will be referred to as general π/M-filters or simpler π/M-filters. In the context of interpolation and decimation, the advantages of the proposed FRM filter classes are as outlined in the following section.

1.1. Advantages of the proposed FRM filter classes and relations to previous work

(1) By making use of the new FRM filter classes, instead of the original one [22], the complexity of the corresponding interpolation/decimation structure can be reduced. The reason is that the efficiency of polyphase decomposition (see Section 3) cannot be fully exploited for cascaded filters unless additional restrictions are imposed. In the original FRM filter class, the overall filter is a combination of parallel and cascade forms (see Section 2) with no additional

restrictions. This implies that one of its three subfilters has to work at the highest of the two sampling rates involved in the interpolation/decimation[4], which makes it less efficient in this context. In the new classes of FRM filters, this problem is circumvented by imposing a relation between the masking filters (according to (3) or (7) in Section 2). In this way, the efficiency of polyphase decomposition can be fully utilized, which means that all filtering will take place at the lowest sampling rate. It should be mentioned that there exist other ways to reduce the complexity emanating from the masking filters, either by relating them to each other [20], [21], [24] or to the model filter [8], [9]. Although those techniques can be utilized for single-rate filtering, they suffer from the same problem as does the original FRM filter class when it comes to polyphase interpolator and decimator realizations.

(2) By making use of the new FRM filter classes instead of the one in [26], which is the only linear-phase FRM FIR filter class considered up to now that fully utilizes polyphase decomposition, one obtains more freedom in selecting the locations of the passband and stopband edges. The reason is that the overall filter's transition band in [26] always includes π/M, implying that the passband (stopband) edge is below (above) π/M. The new general π/M-filter classes do not have this restriction.

(3) A restricted category of π/M-filters is constituted by Mth-band filters whose impulse response values (in the noncausal case) are zero for $n = rM$, $n \neq 0$, r being all integers [36]. Because of this restriction, the transition band of an Mth-band filter always includes π/M, which may or may not be acceptable when the filter is used as an interpolation/decimation filter. When it is acceptable, the use of Mth-band filters can further reduce the complexity of the interpolator/decimator structure as every Mth impulse response value is zero except for the center tap, which equals $1/M$. In addition to being efficient for interpolation/decimation by M, Mth-band filters are also used for other purposes, like the elimination of intersymbol interference in communication systems [36]. The new FRM filter classes can be used to generate Mth-band linear-phase FIR filters for all values of M. This is done by imposing the additional restrictions that either the model filter or the complementary model filter be an Mth-band filter as well. Earlier, only half-band ($M = 2$) FRM linear-phase FIR filters appeared in the literature [38]. (The filters in [26] are not Mth-band filters despite the fact that their transition band includes π/M.) The Mth-band filters in this paper reduce to the half-band filters in [38] for $M = 2$ and can thus be viewed as generalizations of those filters.

Finally, we mention that parts of the work in this paper have been presented at a conference [14] and a workshop [15], but only with regard to one of the classes

[4] It is possible to move all filtering operations to the lowest sampling rate, but then there is a need to introduce multiple instances of the parts that work at the highest sampling rate in the straightforward implementation. That is, the use of such alternatives will not reduce the number of arithmetic operations per sample.

(referred to as Class I in this paper). The other class (referred to as Class II) has not been considered elsewhere.

1.2. Paper outline

Following this introduction, Sections 2 and 3 introduce the new classes of FRM linear-phase FIR filters and the corresponding interpolator/decimator structures, respectively. Section 4 proposes a technique for designing the new filters, and Section 5 provides design examples. Section 6 concludes the paper.

2. Proposed classes of FRM linear-phase FIR filters

This section introduces the new classes of FRM linear-phase FIR filters. In order to be able to more clearly point out the differences between the new classes and existing ones, the section begins with a review of the FRM approach.

2.1. Review of the FRM approach

The basic principle of the FRM approach is to form the overall transfer function according to

$$H(z) = G(z^L)F_0(z) + G_c(z^L)F_1(z) \tag{1}$$

for some positive integer $L > 1$. The filters $G(z)$ and $G_c(z)$ are referred to as a model filter and a complementary model filter, respectively, whereas $F_0(z)$ and $F_1(z)$ are called masking filters. The role of the model filters is to provide several passbands within the interval of 2π with respect to ωT, which here denotes the "digital frequency" variable. This is achieved by replacing z with z^L, which generates the periodic model filter $G(z^L)$ and periodic complementary model filter $G_c(z^L)$. The role of the masking filters is to extract (mask out) one or several passbands of the periodic model filters. In this way, via the use of subfilters with wide transition bands, one can generate a sharp-transition overall filter (see Figure 1). Since the order, and thereby the arithmetic complexity, of direct-form FIR filters, is inversely proportional to the transition bandwidth [10], [11], [19], the use of the FRM approach can reduce the arithmetic complexity substantially for FIR filters with sharp (narrow) transition bands [2], [5], [7], [13], [16], [22], [23], [25]–[27], [31], [34], [35], [38], [42], [43]. The same holds true for approximately linear-phase IIR filters [12], [17], [18], [28].

This paper deals with lowpass filters, in which case typical magnitude responses for the model, masking, and overall filters are as shown in Figure 1. The transition band of $H(z)$ can be selected to be provided by one of the transition bands of either $G(z^L)$ or $G_c(z^L)$. We refer to these two different cases as Case 1 and Case 2, respectively. Further, we let $\omega_c T$, $\omega_s T$, δ_c, and

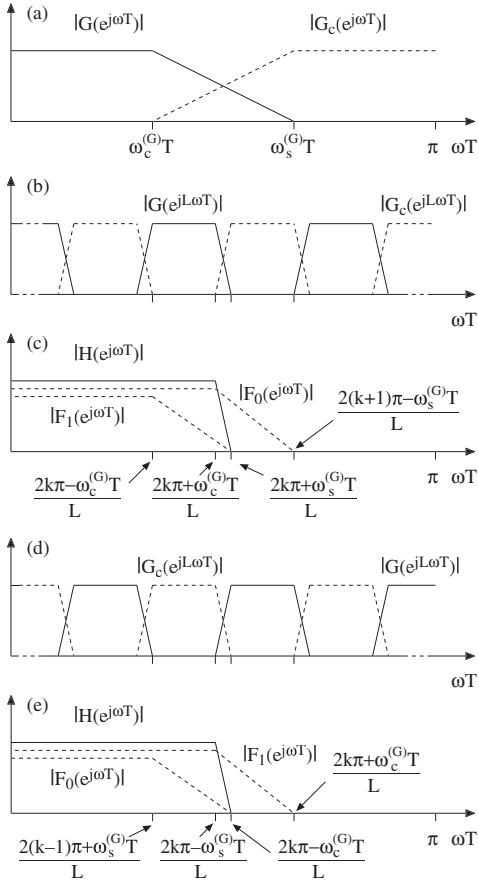

Figure 1. Typical magnitude functions in the FRM approach. (a)–(c): Case 1 design. (a), (d), (e): Case 2 design. The value of k is a positive integer chosen so as to obtain the desired overall passband and stopband edges [22]. For the two proposed classes of FRM filters, k is given by (5) and (8), respectively.

δ_s denote the passband edge, stopband edge, passband ripple, and stopband ripple, respectively, for the overall filter $H(z)$. For the model and masking filters $G(z)$, $G_c(z)$, $F_0(z)$, and $F_1(z)$, additional superscripts (G), (G_c), (F_0), and (F_1), respectively, are included in the corresponding ripples and edges. Finally, note that a special case of FRM filters is obtained by only using the first term on the right-hand side of (1) [32], [36], [37], [39], [40]. The resulting filter $H(z) = G(z^L)F_0(z)$ is a narrowband FRM filter that, in the context of sampling rate conversion, corresponds to two-stage interpolators and decimators and, more generally, multistage interpolators and decimators if $F_0(z)$ is replaced with another narrowband FRM filter, and so on. In the literature, narrowband FRM

filters appeared prior to the general FRM approach, and they have been referred to as interpolated FIR filters. The obvious advantage of the general FRM approach is that it can be used for filters with arbitrary bandwidths, not only narrowband filters. In the context of interpolation and decimation, where the overall filter has a bandwidth of π/M, general FRM filters are thus suitable for smaller values of M, whereas narrowband FRM filters target larger values of M. That is, these two related but different techniques are complementing rather than competing with each other. However, an advantage of the proposed general FRM filter-based converters is that they can be used for arbitrary M, thus including prime numbers, which cannot be handled by narrowband FRM filter-based multistage converters. [5]

2.2. New classes of FRM linear-phase FIR filters

This paper introduces two new classes of linear-phase FRM FIR filters suitable for interpolation and decimation. All filters involved are even-order linear-phase FIR filters. It is therefore possible and convenient to consider noncausal filters. In a practical implementation, the corresponding causal filter is obtained by simply introducing a proper amount of delay. In both of these classes, the noncausal model filter $G(z)$ is a linear-phase FIR filter having a symmetric impulse response $g(n) = g(-n)$ and an even order N_G. The complementary model filter is related to the model filter according to

$$G_c(z) = 1 - G(z), \tag{2}$$

which, together with the assumptions on $G(z)$, implies that $G_c(z)$ is a non-causal linear-phase FIR filter having a symmetric impulse response $g_c(n) = g_c(-n)$ and an even order N_G. Further, in both classes, the masking filters $F_i(z)$, $i = 0, 1$, are linear-phase FIR filters having symmetric impulse responses $f_i(n) = f_i(-n)$ and an even order N_F. These model and masking filters are of the same type as those employed in the original FRM approach for even-order filters [22]. (Alternatively, in the original FRM approach, the masking filters can be odd-order filters, which leads to an odd-order overall filter.) The difference from the original approach is that we will impose a relation between the masking filters in order to make the overall filter suitable for interpolation and decimation. Earlier, such a relation has only been utilized for the special case where $M = 2$ [18], [38]. Relating the masking filters to each other leads to additional restrictions on the feasible values of L and the model filter bandwidth that are not present in the original approach. The details are explained in Sections 2.3 and 2.4, where the overall

[5] For general π/M-filters, multistage structures can also be used in the prime-number case, but then there is a need to include conversion(s) by rational numbers, which may complicate the design and implementation. Furthermore, for the Mth-band filters, multistage structures cannot be used for prime numbers because the use of rational-number converter stages destroys the Mth-band property.

filters are referred to as Class I and Class II filters, respectively. Furthermore, within each class, the overall filter is either a π/M-filter or an Mth-band filter, the latter being obtained by imposing the additional restriction that the model filter (complementary model filter) be a lowpass (highpass) Mth-band filter for Class I (Class II) filters. Altogether, there are consequently four different classes, hereafter called Class I A, I B, II A, and II B, respectively.

2.3. *Class I filters*

For the Class I filters, the masking filters are related according to

$$F_1(z) = F_0(z) - \frac{M}{M-1} F_{00}(z^M) + \frac{1}{M-1}, \tag{3}$$

where $F_{00}(z)$ denotes the 0th polyphase component in the polyphase representation of $F_0(z)$ (see Section 3). The relation in (3) is, of course, not chosen randomly. It is a consequence of the restriction that the overall filter $H(z)$ and the model filter $G(z)$ be Mth-band filters simultaneously, which indicates that this relation can be imposed for general π/M-filters as well. It is interesting to note that, in the special case where $M = 2$, (3) reads

$$F_1(z) = 1 - F_0(-z). \tag{4}$$

In this case, $F_1(z)$ is thus the complement to a frequency-shifted version of $F_0(z)$, and vice versa. This means that $F_1(z)$ $[F_0(z)]$ is a lowpass filter when $F_0(z)$ $[F_1(z)]$ is. For $M = 2$, it has been shown that, provided that L is odd, and the passband and stopband edges of $H(z)$ lie in the neighborhood of $\pi/2$, one can relate the masking filters according to (3) and simultaneously obtain an overall filter with arbitrarily good frequency selectivity by properly designing the model filter and one of the masking filters to meet appropriate lowpass filter specifications [18].

The proposed $M > 2$ case can be viewed as a generalization of the $M = 2$ case, and one may therefore conjecture that it is possible to impose the restriction in (3) and still achieve an arbitrarily good overall filter in terms of frequency selectivity, regardless of the value of M. The situation is however different for $M > 2$, as compared to the $M = 2$ case, in the sense that it is no longer possible to simultaneously satisfy (3) and make both $F_0(z)$ and $F_1(z)$ approximate the lowpass filter [6] responses in Figure 1. This can be concluded experimentally using, e.g., linear programming. However, it has been observed earlier that neither the model filters nor the masking filters need to be conventional frequency-selective filters [28], [41]. It is still possible to obtain an overall lowpass filter. In the proposed Class I filters, the model filters $G(z)$ and $G_c(z)$ and the masking filter $F_1(z)$ are conventional filters, whereas the masking filter $F_0(z)$ possesses additional

[6] It is assumed here that "lowpass filters" ("conventional filters") refer to filters having only one don't-care band, namely the transition band between the passband and stopband.

don't-care bands (transition bands). By properly selecting the periodicity L and bandwidth of the model filter so that these don't-care bands do not overlap with the passbands of the corresponding periodic model filters, one can obtain an overall FRM lowpass filter. The presence of the don't-care bands is due to the relation in (3) as explained below. How to select L and the model filter bandwidth will be discussed afterwards.

From Figure 1 it is seen that both masking filters approximate π/M-filters when the overall filter does so. It is well known that the polyphase component $F_{00}(z)$ of $F_0(z)$ must approximate $1/M$ in the passband and stopband regions of $F_0(z)$ if this filter is to be a linear-phase π/M-filter [1]. When $F_{00}(z) \approx 1/M$ and $F_0(z) \approx 1$ ($F_0(z) \approx 0$), it is seen in (3) that $F_1(z) \approx 1$ ($F_1(z) \approx 0$) as well. Hence, in the frequency regions where both of the masking filters are to have passbands and stopbands, the relation in (3) does not really impose any additional restrictions except that (3) must now be satisfied exactly instead of approximately, as it is automatically in the conventional FRM approach. However, as only one of $F_0(z)$ and $F_1(z)$ is to approximate one (zero) in the remaining part of the overall passband (stopband) region, the relation in (3) imposes restrictions on $F_0(z)$ in its transition band that are not present in a regular lowpass filter design. This means that one has to introduce don't-care bands in the stopband region of $F_0(z)$ as it is not possible to have requirements throughout the whole frequency region. Furthermore, because $F_{00}(z^M)$ is periodic with period $2\pi/M$, the don't-care bands will be centered at $(2q+1)\pi/M$, for $q = 1, 2, \ldots, Q$, where $Q = (M-1)/2$ for M odd and $Q = (M-2)/2$ for M even, and the width of these don't-care bands will be two times that of the transition band of $F_1(z)$, i.e., about $4\omega_c^{(G)}T/L$. Figure 2 shows example frequency responses of the masking filters satisfying (3) for $M = 6$ and Case 2 requirements for passband and stopband edges.

As the Appendix shows, the restriction in (3) implies that the model filter $G(z)$ must be a lowpass π/M-filter in order to avoid overlap between the don't-care bands of the masking filter $F_0(z)$ and the passbands of the periodic filter $G(z^L)$[7]. When both the overall filter $H(z)$ and the model filter $G(z)$ are lowpass π/M-filters, the constant k in the expressions for the passband and stopband edges must satisfy (see Figure 1)

$$k = \begin{cases} \frac{L-1}{2M}, & \text{Case 1} \\ \frac{L+1}{2M}, & \text{Case 2.} \end{cases} \tag{5}$$

[7] This is shown in the Appendix under the assumption that the transition band of the overall filter, and thereby the model filters as well, is zero. In practice, the transition bands are of course nonzero, but this minor deviation from the idealized case can always be compensated for through a proper filter design. That is, the FRM filter classes proposed in this paper can be designed to have arbitrarily good frequency selectivity. However, because the selections of L emanate from the idealized case, there may exist some special cases as to the feasible values of L that are not covered in this paper. It is therefore not claimed that the selections of L in (6) and (9) are strictly necessary in practice.

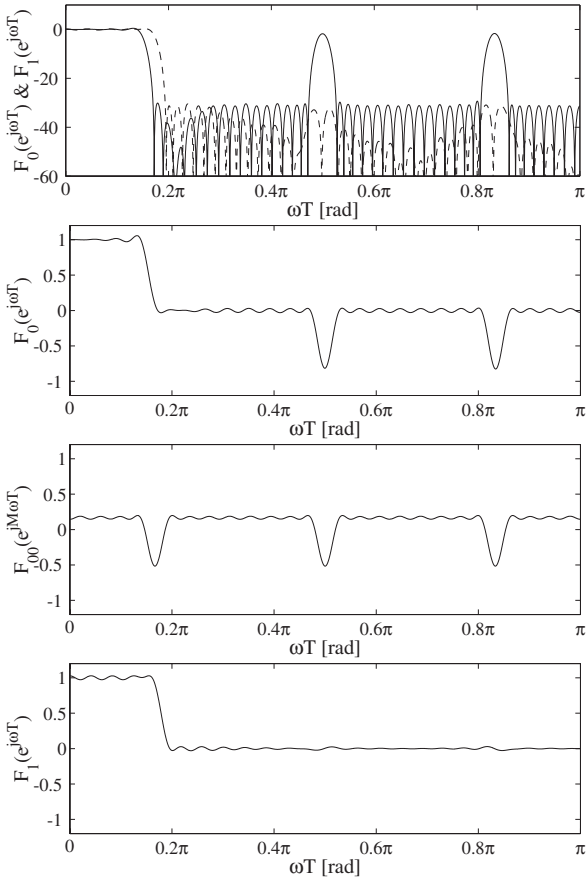

Figure 2. Example of frequency responses of the masking filters satisfying (3) for $M = 6$ and Case 2 requirements for passband and stopband edges. Uppermost figure: Solid line – $F_0(z)$, dashed line– $F_1(z)$.

Thus, because k is an integer, L is chosen according to

$$L = \begin{cases} 2pM + 1, & \text{Case 1} \\ 2pM - 1, & \text{Case 2} \end{cases} \tag{6}$$

for some positive integer p. This means that fewer options are available for selecting the value of L as compared to the original FRM filter class [22], but this is a mild restriction. As will be demonstrated in Section 5, the available choices are sufficient to enable a substantial complexity reduction as compared to the use of conventional direct-form FIR interpolation and decimation filters.

Class I A filters: The model filter in this case is a general linear-phase lowpass π/M-filter with no additional restrictions. This model filter is employed for gen-

erating overall lowpass π/M-filters that are not restricted to include π/M in the transition band.

Class I B filters: The model filter in this case is a lowpass Mth-band filter. This model filter is used for generating overall lowpass Mth-band filters. Recall that the impulse response $h(n)$ of a noncausal Mth-band filter satisfies $h(0) = 1/M$ and $h(n) = 0$ for $n = rM$, r being all integers. This restriction implies that the transition band always includes π/M [36].

2.4. Class II filters

For the Class II filters, the masking filters are related according to

$$F_0(z) = F_1(z) - \frac{M}{M-1} F_{10}(z^M) + \frac{1}{M-1}, \tag{7}$$

where $F_{10}(z)$ denotes the 0th polyphase component in the polyphase representation of $F_1(z)$ (see Section 3). It is noted that (7), like (3), reduces to (4) in the special case where $M = 2$. That is, for $M = 2$, the Class I and II filters coincide. The difference from the relation in (3) is that the roles of the masking filters have been interchanged. That is, $F_1(z)$ is now the filter with don't-care bands. This means that here one must avoid overlap between the don't-care bands of the masking filter $F_1(z)$ and the passbands of the periodic complementary filter $G_c(z^L)$. One can show (in essentially the same way as in the Appendix) that the model filter $G(z)$ here must be a lowpass $(M-1)\pi/M$-filter, i.e., a lowpass filter having a bandwidth of $(M-1)\pi/M$, in order to avoid overlap (see footnote 7). When the overall filter $H(z)$ is a lowpass π/M-filter and the model filter $G(z)$ a lowpass $(M-1)\pi/M$-filter, the constant k in the expressions for the passband and stopband edges must satisfy (see Figure 1)

$$k = \begin{cases} \frac{L-M+1}{2M}, & \text{Case 1} \\ \frac{L+M-1}{2M}, & \text{Case 2.} \end{cases} \tag{8}$$

Thus, because k is an integer, L is chosen according to

$$L = \begin{cases} 2pM + M - 1, & \text{Case 1} \\ 2pM - M + 1, & \text{Case 2} \end{cases} \tag{9}$$

for some positive integer p.

Class II A filters: The model filter in this case is a general linear-phase lowpass $(M-1)\pi/M$-filter with no additional restrictions. When this model filter is employed, one can generate overall lowpass π/M-filters that are not restricted to include π/M in the transition band.

Class II B filters: The model filter in this case is a lowpass $(M-1)\pi/M$-filter satisfying $g(0) = (M-1)/M$ and $g(n) = 0$ for $n = rM$, r being all integers. This model filter is used for generating overall lowpass Mth-band filters. It should be noted that here the complementary model filter is a highpass Mth-band filter.

2.5. Remark on the proposed Mth-band FRM filters

It is not claimed that the proposed ways to realize Mth-band FRM filters (Class I B and II B) are the only ones. Indeed, one may use the original FRM approach and impose the additional restriction that the overall filter be an Mth-band filter. This would lead to a set of nonlinear equations, with the subfilter's coefficients as unknowns, that must be satisfied. Although this technique works in principle, it has at least three drawbacks. First, it would again result in interpolation/decimation structures with unnecessarily high complexity, the reasons being as outlined in the introduction. Second, it would lead to difficulties in the filter design process as it is not trivial to satisfy nonlinear equations in any numerical design procedure. Third, even if one could succeed in designing the filter, it would only be Mth-band with infinite-precision filter coefficients. In a practical implementation, the coefficients must be quantized, which would destroy the Mth-band property as general nonlinear equations cannot be satisfied with finite-precision coefficients. To circumvent all these problems, one should a priori impose appropriate restrictions on the model and masking filters, and this is precisely what has been done for the proposed classes of Mth-band FRM filters.

3. Proposed interpolator and decimator structures

This section introduces the new efficient polyphase interpolator and decimator structures. As polyphase decomposition is key, the section begins with a short review of polyphase interpolators and decimators.

3.1. Review of polyphase interpolator and decimator structures

Interpolation by M consists of M-fold upsampling followed by anti-imaging filtering, whereas decimation by M consists of anti-aliasing filtering followed by M-fold downsampling [3], [6], [44]. To avoid unnecessary computations in the interpolation and decimation processes (multiplications by zero when interpolating; computations of output values that are to be discarded when decimating), it is always beneficial to realize interpolators and decimators using polyphase structures [3], [6], [44], regardless of the filter class that is used. Such interpolator and decimator structures are obtained from the polyphase representation of the transfer function of the interpolation/decimation filter. The polyphase representation of the overall transfer function $H(z)$, for conversions by factors of M, is given by

$$H(z) = \sum_{m=0}^{M-1} z^{-m} H_m(z^M), \tag{10}$$

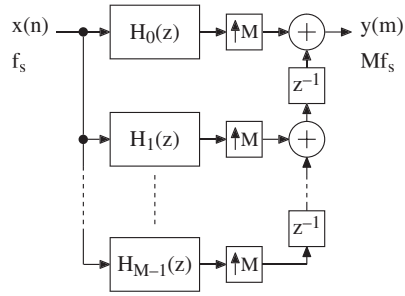

Figure 3. Polyphase interpolator structure.

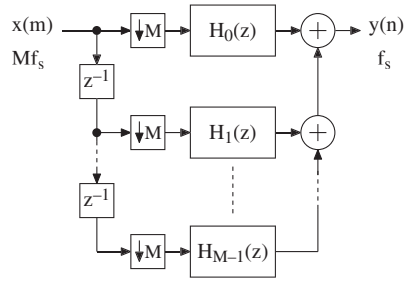

Figure 4. Polyphase decimator structure.

where $H_m(z)$ are referred to as the polyphase components. The corresponding polyphase interpolator and decimator structures are shown in Figures 3 and 4, respectively. The filtering is performed at the lowest sampling rate involved, which results in a low complexity (see footnote 2).

When the polyphase components $H_m(z)$ are independent of each other, the interpolator and decimator are realized straightforwardly, as in Figures 3 and 4. However, depending on the original form of the overall transfer function $H(z)$ (before polyphase decomposition), there may be common parts between the polyphase components. In order to avoid implementing several copies of the same filter part (subfilter), one should therefore identify common parts and utilize this in the final realization. In this way, one can reduce the complexity even further. This is precisely what will be done for the new FRM filter classes in the following section.

3.2. New polyphase interpolator and decimator structures

In accordance with the discussion above, the derivation of the new polyphase interpolator and decimator structures is done in two steps. The first step is to identify the polyphase components $H_m(z)$ in (10). Once these are found, the second step

is to identify common parts between the different polyphase components in order to avoid to implementing several copies of the same subfilter.

In order to identify the polyphase components $H_m(z)$ in (10), the transfer functions $A(z)$ and $B(z)$ are introduced for convenience according to

$$A(z) = \begin{cases} G(z), & \text{Class I filters} \\ 1 - G(z), & \text{Class II filters} \end{cases} \tag{11}$$

and

$$B(z) = \begin{cases} F_0(z), & \text{Class I filters} \\ F_1(z), & \text{Class II filters.} \end{cases} \tag{12}$$

The use of $A(z)$ and $B(z)$ makes it possible to derive and show only one basic structure instead of two (one for each filter class). To do this, we first observe that it follows from the polyphase representation of $A(z)$ that the periodic model filter $A(z^L)$ can be written in polyphase form as

$$A(z^L) = \sum_{m=0}^{M-1} z^{-m} A_m^{(L)}(z^M), \tag{13}$$

where

$$A_0^{(L)}(z) = A_0(z^L) \tag{14}$$

and $A_m^{(L)}(z)$, $m = 1, 2, \ldots, M-1$, are for

Class I filters:

$$A_m^{(L)}(z) = \begin{cases} z^{-2pm} A_m(z^L), & \text{Case 1} \\ z^{-(2p(M-m)-1)} A_{M-m}(z^L), & \text{Case 2} \end{cases} \tag{15}$$

Class II filters:

$$A_m^{(L)}(z) = \begin{cases} z^{-((2p+1)(M-m)-1)} A_{M-m}(z^L), & \text{Case 1} \\ z^{-(2p-1)m} A_m(z^L), & \text{Case 2,} \end{cases} \tag{16}$$

where p is related to L and M through (6) and (9) for Class I and Class II filters, respectively. From (3), (7), (11)–(16), and the polyphase representation of $B(z)$, it now follows that $H(z)$ can be expressed according to the polyphase form in (10) with the polyphase components being

$$H_0(z) = B_0(z) + \frac{(1 - MB_0(z))(1 - A_0^{(L)}(z))}{M - 1} \tag{17}$$

and

$$H_m(z) = B_m(z) + \frac{(1 - MB_0(z))A_m^{(L)}(z)}{M - 1}, \qquad m = 1, 2, \ldots, M-1. \tag{18}$$

From (17) and (18), all polyphase components share the part $[1 - MB_0(z)]/(M-1)$.

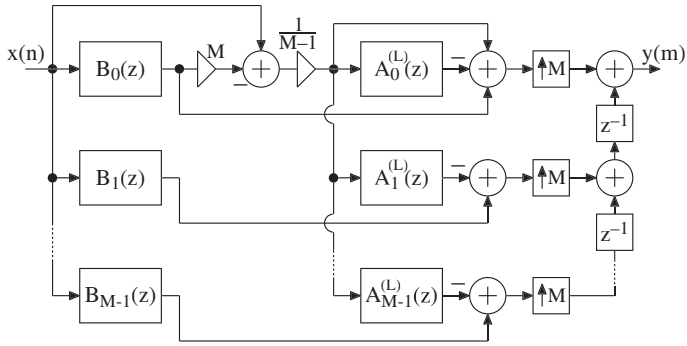

Figure 5. Proposed interpolator structure for the new FRM π/M-filter classes.

Utilizing this fact, one can derive the efficient interpolator structure shown in Figure 5.[8] It is noted that all filtering operations are performed at the lowest sampling rate, and that only one copy of each filter part is needed, yielding a low overall complexity. Hence, using the new classes of FRM filters, the efficiency of polyphase decomposition can be fully exploited.

3.3. New Mth-band polyphase interpolator and decimator structures

As discussed earlier, the new FRM filter classes can also be used for synthesizing Mth-band interpolation and decimation filters. Utilizing the polyphase representation above, this is readily shown as follows. Recall that the impulse response $h(n)$ of a noncausal Mth-band linear-phase FIR filter satisfies

$$h(n) = \begin{cases} 1/M, & n = 0 \\ 0, & n = \pm M, \pm 2M, \ldots \end{cases} . \tag{19}$$

The restriction in (19) implies that the 0th polyphase component in the polyphase representation of the transfer function $H(z)$ is $1/M$, i.e., $H_0(z) = 1/M$ in (10). For the new FRM filter classes, this is achieved by letting $A(z)$ in (11) be an Mth-band filter as well, i.e., by letting $A_0(z) = 1/M$ which corresponds to $G_0(z) = 1/M$ ($G_0(z) = 1 - 1/M$) for Class I (Class II) filters. Indeed, inserting $A_0^{(L)}(z) = A_0(z^L) = 1/M$ into (17), one obtains $H_0(z) = 1/M$. The fact that $A_0^{(L)}(z) = 1/M$ in the Mth-band case means that one obtains further simplifications in the interpolator and decimator structures. This is seen in Figure 6, which shows the new Mth-band polyphase interpolator structure.

[8] As is evident from Figures 3 and 4, the corresponding decimator structure is obtained from the interpolator structure by reversing the signal-flow graph and replacing upsamplers (downsamplers) with downsamplers (upsamplers). For this reason, it is not shown here.

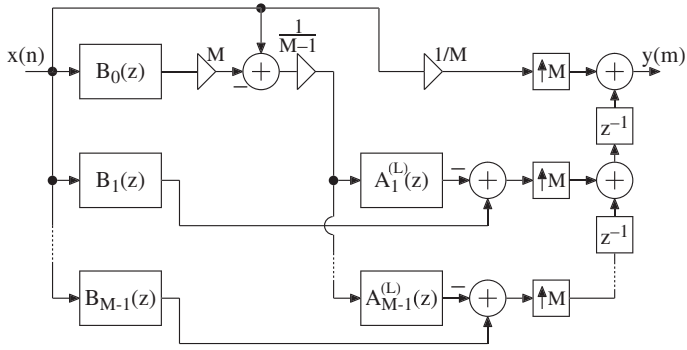

Figure 6. Proposed interpolator structure for the new FRM Mth-band filter classes.

3.4. Multiplication rate

As can be seen in Figures 5 and 6, all polyphase components $B_m(z)$ have the same input, and this is true for all polyphase components $A_m^{(L)}(z)$ as well. This means that it is possible to exploit the coefficient symmetries of the linear-phase model and masking filters, provided that $B_m(z)$ and $A_m^{(L)}(z)$ are realized using transposed FIR filter structures. In this case, the multiplication rate[9], say R, is

$$R = \frac{N_F + N_G + 8}{2M}. \tag{20}$$

In the Mth-band case, the complexity is reduced because one polyphase component of $H(z)$ is a constant, $1/M$. To be precise, R is in this case

$$R = \frac{N_F + N_G - 2\lfloor N_G/(2M)\rfloor + 8}{2M}, \tag{21}$$

where $\lfloor x \rfloor$ stands for the largest integer smaller than or equal to x.

4. Filter design

Three filter design criteria that are frequently used in practice are minimax, least-square, and constrained least-square optimization [36]. Which one to choose depends on the application at hand and is an issue separated from the selection of filter class, if by "filter class" one means the form of the overall transfer function. The main issue of this paper is to introduce the new FRM filter classes for interpolation and decimation and to demonstrate that they offer a low complexity.

[9] The multiplication rate is the number of multiplications per output (input) sample in the interpolator (decimator). For multirate systems, the multiplication rate, not the distinct number of multiplications, is a more appropriate cost measure as it takes into account the data rate at which the multiplications are performed.

The important point then is that the filters to be compared have been designed using the same criterion, not that one has selected an appropriate criterion from the application point of view. For brevity, this paper concentrates on one design criterion, namely minimax, but the other alternatives can naturally be worked out as well. This section introduces a minimax design technique that is suitable for the new filter classes. It is assumed that the following lowpass filter specification is to be satisfied:

$$\begin{aligned} 1 - \delta_c \le H_R(\omega T) \le 1 + \delta_c, \quad & \omega T \in [0, \omega_c T] \\ -\delta_s \le H_R(\omega T) \le \delta_s, \quad & \omega T \in [\omega_s T, \pi], \end{aligned} \tag{22}$$

where $H_R(\omega T)$ denotes the real zero-phase frequency response of $H(z)$.[10]

The overall transfer function $H(z)$ of an FRM filter is expressed in terms of model and masking filters, so the design of any such filter amounts to determining the filter coefficients of these subfilters. Given the periodicity L, as well as the subfilter orders N_G and N_F ($N_F = N_{F0} = N_{F1}$), the overall filter can generally be designed by solving an optimization problem according to: Find the $N_G/2 + N_F + 4$ unknown parameters $g(n), n = 0, 1, \ldots, N_G/2$, $f_0(n), n = 0, 1, \ldots, N_F/2$, $f_1(n), n = 0, 1, \ldots, N_F/2$, and δ_1 that minimize δ_1 subject to

$$\begin{aligned} 1 - \delta_1 \le H_R(\omega T) \le 1 + \delta_1, \quad & \omega T \in [0, \omega_c T] \\ -\delta_1(\delta_s/\delta_c) \le H_R(\omega T) \le \delta_1(\delta_s/\delta_c), \quad & \omega T \in [\omega_s T, \pi], \end{aligned} \tag{23}$$

where $g(n)$, $f_0(n)$, and $f_1(n)$ denote the impulse responses of $G(z)$, $F_0(z)$, and $F_1(z)$, respectively, and

$$H_R(\omega T) = G_R(L\omega T)F_{0R}(\omega T) + [1 - G_R(L\omega T)]F_{1R}(\omega T).$$

The specification in (22) is satisfied if δ_1 after the optimization satisfies $\delta_1 \le \delta_c$. For constrained FRM filters, essentially the same optimization problem is solved. The only difference is that the number of free parameters is reduced. In the proposed filters, $f_0(n)$ and $f_1(n)$ are related via (3) or (7), which means that only $f_0(n)$ or $f_1(n)$ are free parameters.

Due to the fact that $H(z)$ consists of cascaded filters, the optimization problem for FRM filters is nonlinear. One can therefore never guarantee that the obtained solution is globally optimum, but only locally optimum. An important step when designing FRM filters is therefore to find a good start-up solution that subsequently can be improved by further optimization. In that way, one is guaranteed to end up with a good locally optimum solution. In addition, if the start-up solution has been obtained in a "clever way," it turns out that further optimization in many

[10] For a linear-phase FIR filter, having the transfer function $H(z)$, $|H_R(\omega T)| = |H(e^{j\omega T})|$, where $H_R(\omega T)$ and $H(e^{j\omega T})$ denote the real zero-phase frequency response and frequency response, respectively. In the filter design, it is therefore convenient to consider $H_R(\omega T)$. Further, for a noncausal even-order linear-phase FIR filter with a symmetric impulse response $h(n)$, $H_R(\omega T) = H(e^{j\omega T})$.

cases only offers rather modest improvements. To avoid time-consuming and numerically ill-conditioned nonlinear optimization, one may therefore use the start-up solution as the final solution. This may become necessary for very stringent filter requirements, in which case it becomes more difficult to perform nonlinear optimization. One then has to accept a suboptimum solution, but such a solution still has a substantially lower complexity than globally optimum direct-form FIR filters, as will be demonstrated in Section 5. For the further optimization, one may use any general-purpose nonlinear optimization routine [30], or more dedicated ones [28] in order to improve the convergence to speed and/or reliability.

When $G(z)$, $F_0(z)$, and $F_1(z)$ are independent and conventional lowpass filters, a good start-up solution can readily be found by first distributing the overall margins (passband and stopband ripples) over the subfilters $G(z)$, $F_0(z)$, and $F_1(z)$, and then designing these subfilters separately to meet their respective requirements [22]. However, this separate-design approach cannot be used for the proposed classes of FRM filters because of the relations between $F_0(z)$ and $F_1(z)$ imposed by (3) or (7). Therefore, new techniques are introduced below that are suitable for the new classes of FRM filters.

4.1. π/M-filters (Class I A and Class II A filters)

Given the periodicity L, as well as the subfilter orders N_G and N_F ($N_F = N_{F0} = N_{F1}$) a good start-up solution for the π/M-filters is obtained in two steps as follows.

Step 1. Find the $N_G/2 + 2$ unknown parameters $g(n)$, $n = 0, 1, \ldots, N_G/2$, and δ_1 that minimize δ_1 subject to

$$\begin{aligned} 1 - \delta_1 \le G_R(\omega T) \le 1 + \delta_1, &\qquad \omega T \in [0, \omega_c^{(G)} T] \\ -\delta_1 \le G_R(\omega T) \le \delta_1, &\qquad \omega T \in [\omega_s^{(G)} T, \pi] \end{aligned}, \tag{24}$$

where

$$G_R(\omega T) = g(0) + 2 \sum_{n=1}^{N_G/2} g(n) \cos(\omega T n) \tag{25}$$

with $g(n)$ being the impulse response of $G(z)$. Further, the passband and stopband edges of $G(z)$ are

$$\omega_c^{(G)} T = \begin{cases} L\omega_c T - 2k\pi, & \text{Case 1} \\ 2k\pi - L\omega_s T, & \text{Case 2} \end{cases} \tag{26}$$

and

$$\omega_s^{(G)} T = \begin{cases} L\omega_s T - 2k\pi, & \text{Case 1} \\ 2k\pi - L\omega_c T, & \text{Case 2,} \end{cases} \tag{27}$$

respectively, with k being given by (5) and (8) for Class I and Class II filters, respectively.

Step 2. With $G(z)$ obtained in Step 1 fixed, find the $N_F/2+2$ unknown parameters $f_i(n)$, $n = 0, 1, \ldots, N_F/2$, where $i = 0$ ($i = 1$) for Class I (Class II) filters, and δ_2 that minimize δ_2 subject to

$$\begin{array}{rl} 1 - \delta_2 \le H_R(\omega T) \le 1 + \delta_2, & \omega T \in [0, \omega_c T] \\ -\delta_2(\delta_s/\delta_c) \le H_R(\omega T) \le \delta_2(\delta_s/\delta_c), & \omega T \in [\omega_s T, \pi] \end{array}, \tag{28}$$

where

$$H_R(\omega T) = G_R(L\omega T)F_{0R}(\omega T) + [1 - G_R(L\omega T)]F_{1R}(\omega T) \tag{29}$$

with

$$F_{iR}(\omega T) = f_i(0) + 2\sum_{n=1}^{N_F/2} f_i(n)\cos(\omega T n), \tag{30}$$

$$F_{1-i,R}(\omega T) = F_{iR}(\omega T) - \frac{M}{M-1}F_{i0R}(M\omega T) + \frac{1}{M-1}, \tag{31}$$

and

$$F_{i0R}(M\omega T) = f_i(0) + 2\sum_{n=1}^{N_F^{(M)}} f_i(nM)\cos(M\omega T n), \tag{32}$$

where $N_F^{(M)} = \lfloor N_F/(2M) \rfloor$. The optimization problems in Steps 1 and 2 are both convex problems, which guarantees that each solution is optimum in the minimax sense. Each solution can be obtained using standard techniques to this end. For example, by discretizing the problem, and making use of the real rotation theorem [33], the optimum solution (to the discretized problem) can be found by solving a finite-dimensional linear programming problem [30]. This is the technique used later on in the example section. Discretization of the problem can be avoided by using instead semi-infinite programming [4], in which case the optimum solution to the original semi-infinite problems can be obtained. However, provided that the discrete grid is dense enough, the former technique will produce the same solution except for some very small numerical differences that can be neglected in practice. Also note that the filter in Step 1 alternatively can be designed using the fast McClellan-Parks-Rabiner algorithm [29].

The overall filter $H(z)$ satisfies (22) if δ_2 after the above two-step optimization procedure satisfies $\delta_2 \le \delta_c$. When this filter is used as a start-up solution for further nonlinear optimization, the requirements on δ_2 can be relaxed as it will be reduced in the subsequent nonlinear optimization. To minimize the overall complexity for a prescribed acceptable δ_2, one must carefully select L, N_F, and N_G. To this end, the two-step procedure is run for all combinations of L, N_F, and N_G in the neighborhood of estimations of the optimum values of these three parameters. This is a bit of trial and error, but the same problem exists for all classes of FRM filters, and in fact also for conventional FIR filters designed through optimization. Hence, this is not a new problem associated with the proposed filter classes.

For a given L, good estimations of the optimum N_F and N_G are obtained by making use of well-known expressions [10], [11], [19] for estimating the filter orders required for $G(z)$ and $F_{1-i}(z)$, where $i = 0$ ($i = 1$) for Class I (Class II) filters, to meet the lowpass specifications in Figure 1 with passband and stopband ripples of δ_c and δ_s, respectively. [The reason why $F_{1-i}(z)$ is considered here is that this masking filter has more stringent requirements (sharper transition band) than has $F_i(z)$.] Using the same expressions [10], [11], [19], it is also easy to derive the following rough estimation of the optimum L:

$$\hat{L}_{opt} = \sqrt{\frac{2\pi}{M(\omega_s T - \omega_c T)}}. \tag{33}$$

It is also necessary to select δ_1 properly in (24). If δ_1 is too small, then the order of $G(z)$ will be unnecessarily high. If, on the other hand, δ_1 is too large, then there is no margin left for the filters $F_0(z)$ and $F_1(z)$ in the Step 2 optimization, which means that the order of these filters will be very high. It may even turn out that it is impossible to meet the overall specification. To get a good balance between the design margins for $G(z)$ and $F_{1-i}(z)$, it is therefore advisable to select δ_1 about half of $\min(\delta_c, \delta_s)$. The quantity $\min(\delta_c, \delta_s)$ is used because $G_c(z)$ is the complement of $G(z)$, which implies that the passband ripple of $G_c(z)$ equals the stopband ripple of $G(z)$, and vice versa.

The above two-step procedure resembles the first two steps in the iterative scheme introduced in [41], where $F_0(z)$ and $F_1(z)$ are designed simultaneously using linear programming. The main difference is that $F_0(z)$ and $F_1(z)$ are related here according to (3) or (7), whereas in [41] they are independent. In [41], one then continues to design, in an iterative manner, first the pair $F_0(z)$, $F_1(z)$, and then $G(z)$, until the solution between two consecutive iterations is negligible. A key point is that the pair $F_0(z)$, $F_1(z)$ and $G(z)$ take care of different regions of the passband and stopband. The same iterative technique, with the pair $F_0(z)$, $F_1(z)$ and $G(z)$ taking care of the same passband and stopband regions as in [41], cannot be used for the proposed filter classes though. The reason is that the technique in [41] generates masking filters that have quite similar frequency responses. This causes problems in our case because these filters must satisfy (3) or (7). This is why the proposed two-step procedure (followed by further nonlinear optimization) is adopted instead.

4.2. Mth-Band filters (Class I B and Class II B filters)

Because of the restriction in (19), the transition band of an Mth-band filter always includes π/M, and the filter will automatically have a small passband ripple if its stopband ripple is small. To be precise, if the Mth-band linear-phase FIR filter $H(z)$ meets

$$-\delta_s \le H_R(\omega T) \le \delta_s, \qquad \omega T \in [\pi/M + \Delta, \pi], \tag{34}$$

Table 1. Example 1: Class I A filter with $M = 2$, $L = 7$, $\omega_c T = 0.48\pi$, $\omega_s T = 0.5\pi$, $\delta_c = 0.01$, and $\delta_s = 0.01$

	Order	Multiplication rate	Delay
Direct-form filter	194	49	97
Proposed FRM filter	Model filter: 30 Masking filter: 22 Overall filter: 232	14.5	116
Proposed FRM filter without further nonlinear optimization	Model filter: 32 Masking filter: 230 Overall filter: 254	17	127

where $H_R(\omega T)$ again denotes the real zero-phase frequency response of $H(z)$, then it will automatically satisfy

$$1 - \delta_c \le H_R(\omega T) \le 1 + \delta_c, \qquad \omega T \in [0, \pi/M - \Delta] \tag{35}$$

with $\delta_c \le (M-1)\delta_s$ [36]. When designing Mth-band filters, one can therefore for most practical cases concentrate on the stopband region. Because of the additional restrictions in the Mth-band case, Step 1 and Step 2 in Section 4.1 are modified as follows.

Step 1. Find the $N_G/2 - 2\lfloor N_G/(2M)\rfloor + 1$ unknown parameters $g(n), n = 1, 2, \ldots, N_G/2, n \ne rM$, and δ_1 that minimize δ_1 subject to

$$-\delta_1 \le G_R(\omega T) \le \delta_1, \qquad \omega T \in [\pi/M + L\Delta, \pi], \tag{36}$$

where

$$G_R(\omega T) = \frac{1}{M} + 2 \sum_{n=1, n\ne rM}^{N_G/2} g(n) \cos(\omega T n). \tag{37}$$

Step 2. With $G(z)$ obtained in Step 1 fixed, find the $N_F/2+2$ unknown parameters $f_0(n), n = 0, 1, \ldots, N_F/2$, and δ_2 that minimize δ_2 subject to

$$-\delta_2 \le H_R(\omega T) \le \delta_2, \qquad \omega T \in [\pi/M + \Delta, \pi]. \tag{38}$$

Here, the optimum L can be estimated as

$$\hat{L}_{opt} = \sqrt{\frac{\pi}{\Delta \times M}}, \tag{39}$$

which is obtained from (33) by replacing $\omega_c T$ and $\omega_s T$ with $\omega_c T = \pi/M - \Delta$ and $\omega_s T = \pi/M + \Delta$, respectively.

Table 2. Example 2: Class I B filter with $M = 5$, $L = 9$, $\omega_c T = 0.196\pi$, $\omega_s T = 0.204\pi$, and $\delta_s = 0.004$

	Order	Multiplication rate	Delay
Direct-form filter	590	47.4	295
Proposed FRM filter	Model filter: 66 Masking filter: 78 Overall filter: 672	14.2	336

Table 3. Example 3: Class II A filter with $M = 4$, $L = 5$, $\omega_c T = 0.235\pi$, $\omega_s T = 0.25\pi$, $\delta_c = 0.001$, and $\delta_s = 0.00075$

	Order	Multiplication rate	Delay
Direct-form filter	458	57.5	229
Proposed FRM filter	Model filter: 90 Masking filter: 76 Overall filter: 526	21.75	263

5. Design examples

This section illustrates by means of design examples the efficiency of the proposed filters. Three different design examples are considered with details summarized in Tables 1–3, including conventional direct-form FIR filters for comparison[11]. The value of L has been chosen so as to minimize the overall complexity. Figures 7–9 show the filter frequency responses in Example 3. More details and frequency response plots can be found in [14], [15].

Tables 1–3 reveal that the use of the proposed filters can reduce the implementation complexity significantly. The price to pay is a somewhat increased delay, and one should also be aware that the savings of using the proposed filters increase (decrease) with decreasing (increasing) transition bandwidth. However, these features are typical for all FRM FIR filters (see, e.g., [22], [36], [37]), not specific for the proposed ones.

Table 1 includes the results of using the proposed filter without further nonlinear optimization. It is seen that the use of further optimization can reduce the

[11] In Example 2, the order of the conventional Mth-band filters has been estimated by first designing an Mth-band filter with α times wider transition band and then multiplying the so-obtained filter order by α. The reason is that the formulas in [10], [11], [19] cannot be used straightforwardly because the relation between the passband and stopband ripples of Mth-band filters for $M > 2$ is unknown. It is only known that $\delta_c \leq (M-1)\delta_s$ [36]. The reason why estimated orders are used is that numerical problems arise when designing high-order Mth-band minimax filters in MATLAB.

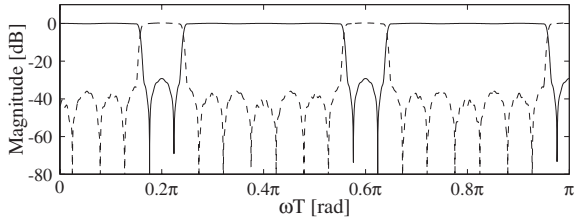

Figure 7. Magnitude response of the periodic model filters in Example 3.

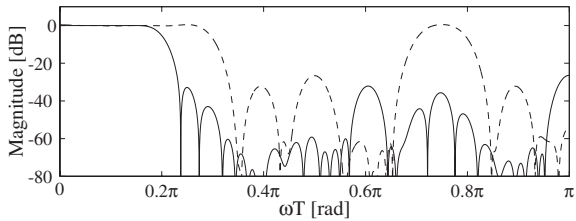

Figure 8. Magnitude response of the masking filters in Example 3.

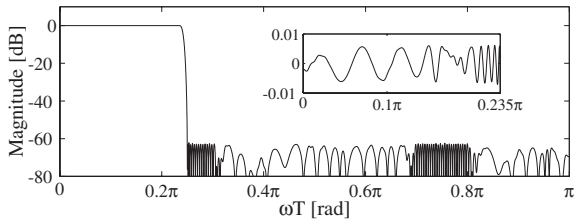

Figure 9. Magnitude response of the overall filter in Example 3.

complexity, but the reduction is rather modest. This illustrates that the design technique introduced in Section 4 generates good filters and start-up solutions for further optimizations. It also illustrates that, even without further optimization, the complexity of the proposed filter is substantially lower than that of the conventional direct-form FIR filter.

6. Conclusion

This paper has introduced two classes of FRM linear-phase FIR sharp-transition filters for interpolation and decimation by arbitrary integer factors M. Both general π/M-filters and Mth-band filters are covered by these classes. As demonstrated through design examples, the new FRM filter classes can be used to generate very efficient interpolator and decimator structures. Compared to previously

existing FRM FIR filter classes for interpolation and decimation, the new ones offer lower implementation complexity and more freedom in selecting the locations of the passband and stopband edges.

Further, the paper introduced design techniques suitable for the new filters, and design examples with $M = 2, 4, 5$ were provided, demonstrating the efficiency of the proposed general π/M-filters as well as the Mth-band filters. The price to pay for the reduced complexity, compared to conventional direct-form linear-phase FIR filters, is a somewhat increased delay. Also note that the savings of using the proposed filters increase (decrease) with decreasing (increasing) transition bandwidths. However, both of these features are typical for all FRM linear-phase FIR filters (see, e.g., [22], [36], [37]), and thus are not specific for the proposed ones.

Appendix

This appendix shows for the Class I filters and a Case 1 design that, under the assumption $\omega_c T = \omega_s T$, which corresponds to $\omega_c^{(G)} T = \omega_s^{(G)} T$, and $\omega_c T = \pi/M$, the bandwidth of $G(z)$ must be $\omega_c^{(G)} T = \pi/M$ in order to avoid overlap between the don't-care bands of the masking filter $F_0(z)$ and the passbands of the periodic filter $G(z^L)$ [Case 2 can be shown similarly]. To prove this, it is first observed that the don't-care bands of $F_0(z)$ occupy $[(2q+1)\pi/M - 2\omega_c^{(G)} T/L, (2q+1)\pi/M + 2\omega_c^{(G)} T/L]$ for $q = 1, 2, \ldots, Q$, where $Q = (M-1)/2$ for M odd and $Q = (M-2)/2$ for M even, and

$$L = (2k + \omega_G)M, \tag{.1}$$

where

$$\omega_G = \omega_c^{(G)} T/\pi = p/M, \quad p \in [1, 2, \ldots, M-1], \tag{.2}$$

whereas the passbands of $G(z^L)$ occupy $[2\pi r/L - \omega_c^{(G)} T/L, 2\pi r/L + \omega_c^{(G)} T/L]$ for $r = 1, 2, \ldots, R$, where $R = kM + (p-1)/2$ for p odd and $R = kM + p/2$ for p even. To ensure that the don't-care bands and passbands do not overlap each other, it now follows that the inequality

$$|2\pi r/L - (2q+1)\pi/M| \geq 3\omega_c^{(G)} T/L \tag{.3}$$

must be satisfied for all pairs of $r = 1, 2, \ldots, R$ and $q = 1, 2, \ldots, Q$. Utilizing (40) and (41), (42) can equivalently be written as

$$|2(r - k(2q+1)) - \frac{p(2q+1)}{M}| \geq \frac{3p}{M}. \tag{.4}$$

For $r \leq k(2q+1)$, it is obvious that (43) is satisfied, regardless of the value of p. Furthermore, it is apparent that (43) holds for $p = 1$, i.e., $\omega_G = 1/M$, and $M > 2$ also when $r > k(2q+1)$ for all values of q because $2M - (2q+1) \geq M \geq 3$ for $M > 2$. (Recall that the $M = 2$ (corresponding to $Q = 0$) case need not be

investigated because the masking filter $F_0(z)$ does not possess don't-care bands in this case; see Section 2.3.) This proves that $\omega_G = 1/M$ is a sufficient condition to avoid overlap. To prove that $\omega_G = 1/M$ is also a necessary condition, it suffices to show for $p > 1$ that (a) $R - kQ > 0$ (because then there will always be a value of r such that $r - k(2q + 1) = 1$) and (b) there exists a value of q such that

$$-\frac{3p}{M} < 2 - \frac{p(2q+1)}{M} < \frac{3p}{M}. \tag{.5}$$

From the values of Q and R given above, it is seen that condition (a) is satisfied for $1 < p < M$. As for condition (b), it is noted that (44) can be restated as

$$\frac{2M}{p} - 3 < 2q + 1 < \frac{2M}{p} + 3. \tag{.6}$$

It is readily verified that for $1 < p < M$ and $M > 2$ one always can find a $q \in [1, 2, \ldots, Q]$ satisfying this condition. This concludes the proof. Finally, it is also noted that (45) cannot be satisfied with $p = 1$, $M > 2$, and $q \in [1, 2, \ldots, Q]$, which is in line with the fact that the selection $\omega_G = 1/M$ ensures that (43) is satisfied.

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