THREE CLASSES OF IIR COMPLEMENTARY FILTER PAIRS WITH AN ADJUSTABLE CROSSOVER FREQENCY

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ABSTRACT

Three classes of complementary recursive low-pass/high-pass filter pairs are introduced. For each class, the crossover frequency can be arbitrarily selected without changing the predetermined stop-band attenuation being the same for both filters and the filter pair is constructed using two all-pass sub-filters as building blocks. Based on the properties of elliptic minimal Q-factors transfer functions, simple expressions are derived for evaluating the coefficients for the all-pass sections to arbitrarily change the crossover frequency. Design procedures are developed for synthesizing filter pairs implemented as a parallel connection of two all-pass sub-filters and for two classes of filter pairs constructed as tapped cascaded interconnections of two identical all-pass sub-filters. The first of them together with the direct parallel connection of two all-pass filters provide the power-complementary property, whereas the second one provides the magnitude-complementary property.

1. INTRODUCTION

Complementary infinite-impulse response (IIR) filter pairs can be synthesized for generating power-complementary, all-pass complementary, or magnitude-complementary filter pairs (see, e.g., [1]). A very attractive alternative to generate a power-complementary low-pass/high-pass IIR filter pair is to use lattice wave digital filters (parallel connections of two all-pass filters) [2]–[4]. Usually, the crossover frequency for this filter pair occurs in the middle of the base-band, that is, it is located exactly at 1/4 in terms of the normalized frequency. If the stop-band attenuation of both filters is the same, then the low-pass and high-pass filters are both half-band IIR filters (see, e.g., [4], [5]).

This paper proposes a technique to change, by means of simple formulae, the location of the crossover frequency of the half-band filter pair to an arbitrary location while still retaining the attenuation properties of the initial half-band filters. For this purpose, elliptic minimal Q-factors (EMQF) transfer functions introduced in [6], [7] provide directly the desired solution.

In addition to the direct parallel connection of two all-pass filters, two structures constructed as tapped cascaded interconnections of two identical all-pass filters are considered based on the use of the synthesis schemes described in [8]–[10]. The first (second) structure allows one to generate power-complementary (magnitude-complementary) filter pairs with an adjustable crossover frequency. The advantage of using tapped cascaded interconnections of two identical all-pass sub-filters is fact that that the all-pass sub-filters are of very low orders.

2. USE OF EMQF FILTERS

This section shows how the properties of EMQF filters [6], [7] can be exploited in a very straightforward manner for generating complementary IIR filter pairs with an adjustable crossover frequency such that the stop-band attenuation of both filters remains the same. Due to the fact that EMQF filters comprise a half-band filter as a special case, we start with a half-band filter pair to generate the desired complementary filter pair whose crossover frequency can be arbitrarily chosen.

2.1 Properties of EMQF filters for Generating Power-Complementary Filter Pairs

A complementary IIR filter pair constructed as a parallel connection of two all-pass sub-filters is shown in Fig. 1. For half-band IIR filters, this filter pair is given by (see, e.g., [4], [5])

$$\mathbf{G}^{HB}(z)=\frac{1}{2}\left[A_0^{HB}(z)\pm A_1^{HB}(z)\right]=\frac{1}{2}\left[\prod_{i=2,4,\ldots}^{(N+1)/2}\frac{\beta_i^{HB}+z^{-2}}{1+\beta_i^{HB}z^{-2}}\pm z^{-1}\prod_{i=3,5,\ldots}^{(N+1)/2}\frac{\beta_i^{HB}+z^{-2}}{1+\beta_i^{HB}z^{-2}}\right],\ \beta_i^{HB}<\beta_{i+1}^{HB}. \tag{1}$$

Here, N, the filter orders, is restricted to be odd and $\mathbf{G}^{HB}(z) = [\,G_{LP}^{HB}(z)\ G_{HP}^{HB}(z)\,]^T$ is a vector containing the low-pass (with the plus sign) and high-pass (with the minus sign) half-band filter transfer functions $G_{LP}^{HB}(z)$ and $G_{HP}^{HB}(z)$. The subscript 'HP' is used to emphasize that the filters are half-band filters. One of poles of these transfer functions is located at the origin and the remaining poles are complex-conjugate pairs being located on the imaginary axis at $z=\pm jr_i$ for $i=2,3,\ \cdots,(N+1)/2$, giving $\beta_i^{HB}=(r_i)^2$. The notation $\beta_i^{HB}<\beta_{i+1}^{HB}$ in Eq. (1) indicates how the poles are shared between the all-pass sections.

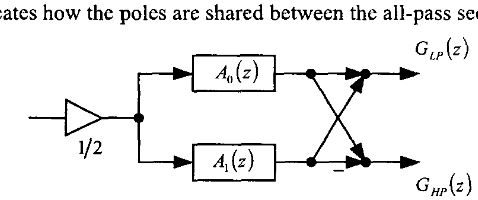

Figure 1. Double-complementary IIR filter pair constructed as a parallel connection of two all-pass filters.

This filter pair when generated by means of an odd-order elliptic filter is characterized by the following attractive properties. First, the sum of the squared-magnitude responses of the two filters is identically equal to unity. Hence, the filter pair has the power-complementary property. Second, the 3-dB crossover frequency (the frequency where the squared-magnitude responses of both $G_{LP}^{HB}(z)$ and $G_{HP}^{HB}(z)$ achieve the value of 1/2) is located at $f_{3dB}=1/4$ in terms of the normalized frequency. Third, the pass-band and stop-band edges, denoted by f_p and f_s in terms of the normalized frequency, satisfy $f_s=1/2-f_p$. Fourth, the attenuation of both the low-pass and high-pass filters is the same. Since the sum of $G_{LP}^{HB}(z)$ and $G_{HP}^{HB}(z)$ is $A_0^{HB}(z)$, this filter pair is also an all-pass complementary filter pair, thereby called as a double-complementary filter pair [1].

The low-pass/high-pass EMQF filter pair is generated with the aid of the above half-band filter pair as follows [6], [7]:

$$\mathbf{G}(z)=\frac{1}{2}\left[A_0(z)\pm A_1(z)\right]=\frac{1}{2}\left[\prod_{i=2,4,\ldots}^{(N+1)/2}\frac{\beta_i+\alpha(1+\beta_i)z^{-1}+z^{-2}}{1+\alpha(1+\beta_i)z^{-1}+\beta_i z^{-2}}\right.$$
$$\left.\pm\frac{\alpha_1+z^{-1}}{1+\alpha_1 z^{-1}}\prod_{i=3,5,\ldots}^{(N+1)/2}\frac{\beta_i+\alpha(1+\beta_i)z^{-1}+z^{-2}}{1+\alpha(1+\beta_i)z^{-1}+\beta_i z^{-2}}\right],\ \beta_i<\beta_{i+1},\tag{2}$$

where $\mathbf{G}^{HB}(z)=[\,G_{LP}^{HB}(z)\ G_{HP}^{HB}(z)\,]^T$ is again a vector containing the resulting low-pass and high-pass filter transfer functions $G_{LP}(z)$ and $G_{HP}(z)$ that can be still implemented as shown in Fig. 1. The very simple formulae for converting the delay term z^{-1} and the second-order all-pass sections in Eq. (1) to the first-order and the second-order all-pass sections in Eq. (2) will be developed in the following subsection.

The properties of EMQF filters guarantee that for the EMQF filter pair, as given by Eq. (2), the crossover frequency can be changed while keeping the attenuation of both the low-pass filter and high-pass filter the same as for the start-up half-band filter pair. Second, for these filter pairs, the z-plane poles are on the circle that is orthogonal with the unit circle and centered on the real axis [6], [7]. When the center of the circle approaches infinity, the circle degenerates into the imaginary axis, and the half-band filter with the poles on the imaginary axis is obtained. This property provides a very straightforward approach to transforming a half-band filter pair to an EMQF filter pair whose crossover frequency can be arbitrarily chosen, as will be described in the following subsection.

2.2 Frequency transformations

This subsection shows how an EMQF filter with an arbitrary 3-dB cutoff frequency f_{3dB} can be generated with the aid of a prototype half-band filter and the following bilinear transform:

$$s=k(z-1)/(z+1)\,,\tag{3}$$

where k is a constant to be adjusted.

Figure 2 illustrates the overall transformation procedure. The start-up filter is an odd-order half-band filter with the selectivity factor ξ defined by

$$\xi=\tan(\pi f_s)/\tan(\pi f_p),\tag{4}$$

where $f_s=1/2-f_p$. The z-plane poles of the half-band filter poles are located on the imaginary axis and the 3-dB cutoff frequency is located at $z_{3dB}^{HB}=e^{j\pi/2}$, as illustrated in Fig. 2(a) for a third-order half-band filter. Then, the inverse bilinear transform with $k=\sqrt{\xi}$ is used to map this half-band filter from the z-plane to the s-plane to generate the corresponding analogue prototype filter, as shown in Fig. 2(b). The resulting s-plane poles of the analogue prototype are located on the circle centered at the origin with the radius equal to $k=\sqrt{\xi}$ and the 3-dB cutoff frequency being located at $s_{3dB}=j\sqrt{\xi}$ [6]. The analogue prototype is then mapped from the s-plane the to the z-plane by using $k=\sqrt{\xi}/\tan(\pi f_{3dB})$ in Eq. (3) resulting in the EMQF filter whose 3-dB cutoff frequency is placed at the desired location $z_{3dB}=e^{j2\pi f_{3dB}}$, as shown in Fig. 2(c). Since the start-up half-band filter and resulting EMQF filter have the common analogue prototype, the double-complementary property of the half-band filter is preserved when using these transformations.

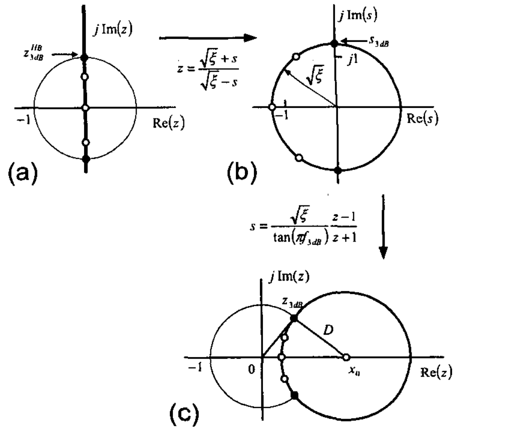

Figure 2. Frequency transformations. (a) Start-up half-band filter. (b) Corresponding analogue prototype filter. (c) EMQF filter with the desired 3-dB cutoff frequency at $z_{3dB}=e^{j2\pi f_{3dB}}$.

The key role in the mappings of Fig. 2 is the bilinear transform, as given by Eq. (3). The value of the constant k is first adjusted to map the digital half-band filter to the analogue prototype, and adjusted again to map the analogue prototype to the EMQF filter with the desired f_{3dB}. The goal is to express the constants α_1, α, and β_i of the EMQF filter [see Eq. (2)] in terms of the constants β_i^{HB} of the start-up half-band filter [see Eq. (1)], and the desired value of f_{3dB}. This is achieved when the transformations of Fig. 2 are directly applied to the individual all-pass sections in Eqs. (1) and (2). Table 1 summarizes the formulae for converting the start-up half-band filter to the EMQF filter with the given 3-dB cutoff frequency [6], [7]. The normalized pass-band and stop-band edges for the resulting EMQF filter are given by

$$f_p=\frac{1}{\pi}\tan^{-1}\left(\tan(\pi f_{3dB})/\sqrt{\xi}\right)\ \text{and}\ f_s=\frac{1}{\pi}\tan^{-1}\left(\sqrt{\xi}\tan(\pi f_{3dB})\right).\tag{5}$$

It should be pointed out that the 3-dB cutoff frequency of a single filter is equivalent to the 3-dB crossover frequency of a complementary filter pair.

Table 1: Parameters for a power-complementary filter pair with the given 3-dB cutoff frequency f_{3dB}

First-order Section	$\alpha_1=\left(1-\tan(\pi f_{3dB})\right)/\left(1+\tan(\pi f_{3dB})\right)$	
Second-order Section	$\alpha=\dfrac{1-(\tan(\pi f_{3dB}))^2}{1+(\tan(\pi f_{3dB}))^2}$	$\beta_i=\dfrac{\beta_i^{HB}+\alpha_1^2}{\beta_i^{HB}\alpha_1^2+1}$

3. SYNTHESIS OF FILTER PAIRS

This section shows how exploit the properties of the EMQF filters for synthesizing complementary IIR low-pass/high-pass filter pairs with an adjustable crossover frequency for the three filter pair classes mentioned in Introduction.

3.1 Parallel connection of two all-pass sub-filters

Consider an IIR power-complementary filter pair as depicted in Fig. 1. Given the odd order N, the 3-dB crossover frequency f_{3dB}, and the minimum stop-band attenuation A_s in decibels, the synthesis can be carried out as follows:

1) Design the start-up power-complementary IIR half-band filter pair, as given by Eq. (1), such that the minimum stop-band attenuation is exactly A_s.[1] Let the resulting pass-band and stop-band edges be f_p and $f_s = 1/2 - f_p$.
2) Design the power-complementary IIR filter pair, as given by Eq. (2), having the desired 3-dB crossover frequency f_{3dB} by determining its parameters using the formulae of Table 1.
3) Determine the pass-band and stop-band edges of the resulting filter pair according to Eqs. (4) and (5).

Figure 3 shows responses of some filter pairs for $N = 7$ and $A_s = 60$ dB. The thick line shows the response of the half-band filter pair. To obtain a new filter pair, only 5 constant values have to be computed using the formulae of Table 1.

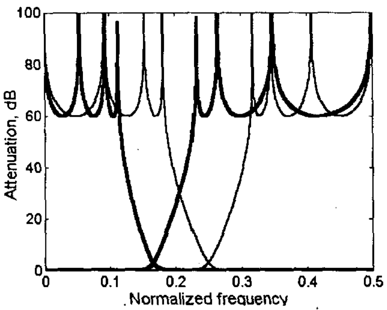

Figure 3. Responses for power-complementary seventh-order filter pairs. Thick-line: f_{3dB} =1/4. Thin line: f_{3dB} =1/6.

3.2 Tapped cascaded interconnection of two all-pass filters having the power-complementary property

For a power-complementary filter pair constructed as a tapped cascaded interconnection of two identical all-pass filters, the low-pass and high-pass transfer functions are given by [8], [10]

$$H_{LP}(z) = \sum_{m=0}^{M} a[m][A_0(z)]^m [A_1(z)]^{M-m} \qquad (7a)$$

$$H_{HP}(z) = \sum_{m=0}^{M} (-1)^m a[M-m][A_0(z)]^m [A_1(z)]^{M-m}, \qquad (7b)$$

where M is an odd integer and $A_0(z)$ and $A_1(z)$ are the all-pass filters in Eq. (1) or Eq. (2). An efficient implementation form for the above filter pair is show in Fig. 4 [10] [2]. The details on how to convert the $a[m]$'s to the k_l 's can be found in [10].

Given the odd orders M and N, the 3-dB crossover frequency f_{3dB}, and the minimum stop-band attenuation A_s, the optimized filter pair can be generated as follows:

1) Determine $\Delta = 10^{-A_s/10}$.
2) Optimize the $a[m]$'s to maximize θ_p such that the amplitude response of a nonlinear-phase transfer function $F(z) = \sum_{m=0}^{M} a[m] z^{-m}$ stays within the limits 1 and $\sqrt{1-\Delta}$ in the normalized pass-band region [0, θ_p] and the maximum amplitude value is $\sqrt{\Delta}$ in the normalized stop-band region [1/2−θ_p, 1/2].
3) Perform the synthesis scheme of Subsection 3.1 with the main exception that now the desired stop-band attenuation for the start-up half-band filter pair, as given by Eq. (1) and determining the all-pass sections in Fig. 4, is given by

$$\hat{A}_s = -10\log_{10}\{\cos[\pi(1/2-\theta)]\}. \qquad (8)$$

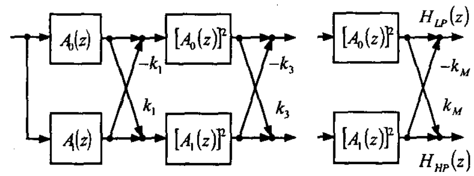

Figure 4. Lattice structure for the proposed power-complementary filter pair.

As shown [8]–[10], the basic idea of using tapped cascaded interconnections of two identical all-pass sub-filters is based on following two facts. First, the pass-band and stop-band regions for the direct parallel connections and the overall structures are the same. Second, due to use of several copies of two all-pass sections, the requirements for the parallel connection of the all-pass sections become significantly milder compared with the overall filter. This enables one to use only first-order and second-order all-pass sections, thereby making the tuning of the crossover frequency of the filter pair extremely simple without changing the tap coefficients combining the two all-pass sections. Step 2 in the above procedure can be performed by slightly modifying the design scheme described in [8].

Figure 5 shows responses of the some power-complementary filter pairs in the case where A_s = 60 dB and M = 5 identical copies of two all-pass filters of orders 2 and 1 are used, that is, N = 3. For this design, $\hat{A}_s$ = 19.37 dB and the new filter pair is generated by evaluating only 3 constant values according to the formulae of Table 1.

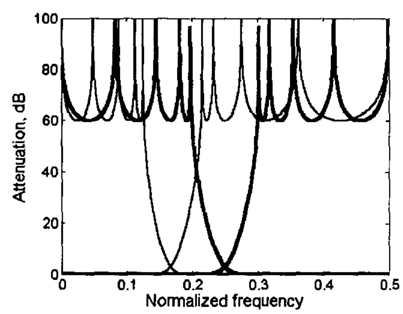

Figure 5. Power-complementary filter pairs constructed using 5 identical copies of the first-order and the second-order all-pass filters. Thick-line: f_{3dB} =1/4. Thin line: f_{3dB} =1/6.

3.3 Tapped cascaded interconnection of two all-pass filters having the magnitude-complementary property

For a magnitude-complementary filter pair, the low-pass and high-pass transfer functions are given by [9]

$$H_{LP}(z) = \sum_{m=0}^{M} a[m][A_0(z)]^m [A_1(z)]^{M-m} \qquad (9a)$$

[1] At this stage, $A_0(z) \equiv A_0^{HP}(z)$ and $A_1(z) \equiv A_1^{HP}(z)$ in Fig. 1 (see Eq. 1).

[2] It should be pointed out that this implementation form is valid only for the power-complementary filter pairs under consideration in this subsection. The structure of Fig. 4 has been derived based on Fig. 11 in [10] by removing the decimation by a factor of two. Note that the subscripts and signs of the k's are different.

$$H_{HP}(z)=\sum_{m=0}^{M}b[m][A_0(z)]^m[A_1(z)]^{M-m} \quad (9b)$$

Here, M is an integer being two times an odd integer, $a[M-m]=a[m]$, $b[M-m]=b[m]$, and $b[m]=-a[m]$ for $m=0,1,\cdots,M/2-1$. Furthermore, $a[M/2]=b[M/2]=1/2$ and $a[m]=b[m]=0$ for the remaining odd values of m. An efficient implementation for the above magnitude-complementary filter pair is shown in Fig. 6 [9]. The sum of the above two transfer functions is $[A_0(z)]^{M/2}[A_1(z)]^{M/2}$ so that they form also an all-pass complementary filter pair. It should be noted that for the above filter pair, the magnitude responses achieve the value of 1/2 (−6 dB) at the crossover frequency.

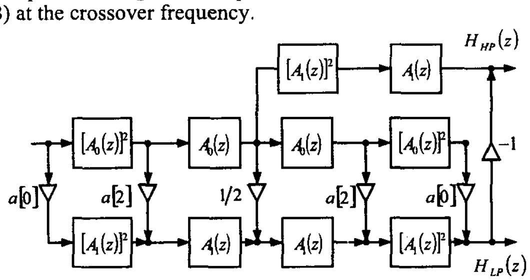

Figure 6. Structure for the proposed magnitude-complementary (all-pass complementary) filter pair. $M=6$.

In this case, the overall synthesis can be performed as in Subsection 3.2 by replacing Steps 1 and 2 by

1) Determine $\Delta=10^{-A_s/20}$.
2) Optimize the $a[m]$'s to maximize θ_p such that the zero-phase frequency response of the FIR filter transfer function [3] $F(z)=z^{-M/2}\{1/2+\sum_{n=1}^{(M+2)/4}a[M/2-(2n-1)][z^{2n-1}+z^{-(2n-1)}]\}$ stays within the limits 1 and $1-\Delta$ in the normalized pass-band region [0, θ_p] and within the limits 0 and Δ in the normalized stop-band region [1/2−θ_p, 1/2].

Figure 7 shows responses of the some filter pairs in the case where A_s = 60 dB and M = 6 identical copies of two all-pass filters of orders 2 and 1 are used, that is, N = 3. For this design, $\hat{A}_s$ = 14.53 dB. To generate a new filter pair, only 3 constant values have to be computed using the formulae of Table 1.

4. CONCLUSION

This paper has introduced three design approaches for generating complementary low-pass/high-pass IIR filter pairs with an adjustable crossover frequency. The start-up complementary filters are low-pass/high-pass half-band filter pairs constructed as a parallel connection of two all-pass filter sections (a lattice wave digital filter). Exploiting the properties of elliptic minimal Q-factors transfer functions, simple formulae for the direct calculation of the coefficients have been derived. This allows one to implement programmable complementary filter pairs with an adjustable crossover frequency in a very simple and straightforward manner. For this purpose, future work is devoted to generate simplified forms for the formulae of Table 1. The errors caused by these new formulas can be compensated by slightly over-designing the start-up filter pair so that the filter pair with the desired crossover frequency still meets the given criteria.

Particular benefits have been obtained for the realization structures based on the use of the tapped cascaded interconnection of two identical all-pass sub-filters leading to power-complementary or magnitude-complementary filter pairs. For these structures, the all-pass sub-filters are of a very low order, thereby making the adjustment of the crossover frequency significantly easier.

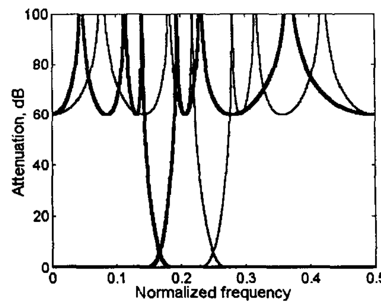

Figure 7. Magnitude-complementary filter pairs constructed using 6 identical copies of the first-order and the second order all-pass filters. Thick-line: f_{3dB} =1/4. Thin line: f_{3dB} =1/6.

5. ACKNOWLEDGEMENTS

This work was supported by the Academy of Finland, project No. 44876 (Finnish Centre of Excellence Program, 2000–2005). Authors greatly appreciate the inspiring discussions with Prof. Markku Renfors, Tampere University of Technology.

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[3] This $F(z)$ is a transfer function of a linear-phase FIR half-band filter. The zero-phase frequency response is obtained by omitting the phase term and is expressible as $F(\omega)=1/2+2\sum_{n=1}^{(M+2)/4}a[M/2-(2n-1)]\cos[(2n-1)\omega]$.