Approximately Linear-Phase Recursive Digital Filters with Variable Magnitude Characteristics

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Abstract— **This paper considers designing in the minimax sense complementary low-pass/high-pass approximately linear-phase recursive filters with variable magnitude characteristics. A filter structure based on the parallel connection of a delay and a variable fractional delay all-pass filter is proposed for implementing these filters. The filter optimization is performed in two basic steps. First, an initial filter is generated using a simple design scheme. Second, this filter is used as a start-up solution for further optimization being carried out by an efficient constrained nonlinear optimization algorithm. Examples are included for illustrating the efficiency of the proposed design scheme. In addition, the performance and the complexity of the proposed variable recursive digital filters are compared with those of the other variable recursive digital filters proposed in the literature. This comparison shows that the number of multipliers for the proposed filters is less than 30 percent compared with the other existing structures.**

I. Introduction

IN VARIOUS digital signal processing applications, there is a need for filter with variable frequency characteristics. These applications include, e.g., sampling rate conversion, echo cancellation, phased-array antenna systems, time delay estimation, timing adjustment in all-digital receivers, modeling of music instruments, and speech coding and synthesis [1]–[4]. Recently, research on the optimal design and the efficient implementation of the variable fractional delay filters has received considerable attention [5]–[8]. However, the implementation of filters with variable magnitude characteristics have not gained as much attention and have been considered only by a few authors [9], [10]. The purpose of this contribution is to propose a new class of magnitude-selective variable digital filters and an algorithm for their optimization.

The methods for designing variable digital filters can be broadly classified into two categories, namely, frequency transformation methods [11]–[14] and spectral parameter approximation methods [3], [5]–[10], [15], [16]. The disadvantage of the methods, belonging into the former class, is that the edge frequencies and the ripples of the various bands cannot be independently controlled. The second class of filters does not suffer from this restriction. In this technique, the coefficients of the variable filter are expressed as the polynomials of the adjustable parameter defining the desired filter characteristic.

Variable digital filters can be constructed using either finite impulse-response or infinite impulse-response filters. From the implementation point of view, one of the best structures for recursive digital filters is a parallel connection of two all-pass filter sections. These structures have some advantageous properties, such as a reasonably low coefficient sensitivity and a low roundoff noise level. The main drawback of these recursive filters is that their phase response is inherently very nonlinear in the passband. However, there exist two straightforward approaches to linearizing the phase in the passband. The first one is to cascade the nonlinear recursive filter with a phase equalizer which is also an all-pass filter. The second one is to select one of the branches to be a pure delay term [17], [18]. In this case, the phase of the other branch is forced to follow closely the linear phase in the passband in order to provide a small passband variation for the overall filter. Simultaneously, the overall phase response becomes rather linear in the passband.

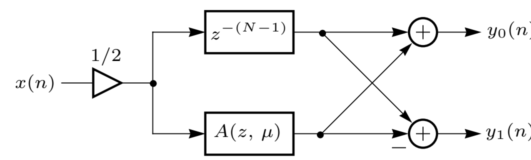

Fig. 1. Complementary low-pass/high-pass approximately linear-phase variable recursive filter pair implemented as a parallel connection of a delay and a variable fractional delay all-pass filter.

This contribution proposes an efficient structure for implementing magnitude-selective approximately linear-phase variable recursive digital filters. This filter structure is based on the parallel connection of a delay and a variable fractional delay all-pass filter. In addition, an algorithm is proposed for optimizing the magnitude response of these filters. Furthermore, the performance and the complexity of these filters are compared with some other variable recursive digital filters proposed in the literature showing that the number of multipliers for the proposed filters are less than 30 percent compared with the other existing structures.

II. Variable Recursive Digital Filters

The transfer functions of the approximately linear-phase variable digital filter pair as shown in Fig. 1 are given by

$$H_{0,1}(z,\ \mu) = \frac{1}{2}\left[z^{-(N-1)} \pm A(z,\ \mu)\right]. \tag{1}$$

Here, $H_0(z,\ \mu)$ with the plus sign and $H_1(z,\ \mu)$ with the minus sign are the low-pass and high-pass filters, respectively, and $A(z,\ \mu)$ is a variable fractional delay all-pass filter [5], [6], [8], [19] of order N and is expressible as

$$A(z,\ \mu) = \frac{z^{-N}C(z^{-1},\ \mu)}{C(z,\ \mu)}, \tag{2a}$$

where

$$C(z,\ \mu) = 1 + \sum_{n=1}^{N} a_n(\mu)z^{-n} = 1 + \sum_{n=1}^{N}\left[\sum_{p=0}^{P} c_{pn}\mu^p\right]z^{-n}. \tag{2b}$$

Here, μ is an adjustable parameter in the range $[-1,\ 1]$ and each coefficient in the overall filter is given as a polynomial of degree P in μ. Figure 2 shows an efficient implementation for the variable fractional delay all-pass filter based on the so-called gathering structure [5].

The phase response of the variable fractional delay all-pass filter $A(z,\ \mu)$ is expressible as

$$\Theta_A(\omega,\ \mu) = -N\omega + 2\arctan\left[\frac{\sum_{n=1}^{N} a_n(\mu)\sin n\omega}{1+\sum_{n=1}^{N} a_n(\mu)\cos n\omega}\right], \tag{3}$$

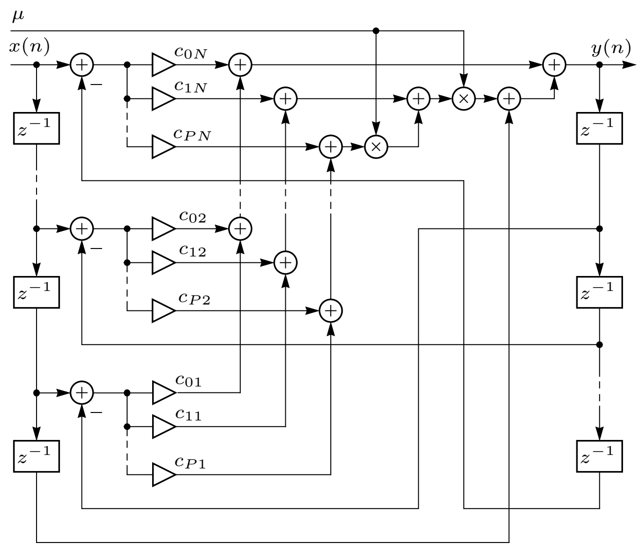

Fig. 2. Gathering structure for the variable fractional delay all-pass filters.

whereas the magnitude responses for $H_0(z,\,\mu)$ and $H_1(z,\,\mu)$ can be written as

$$|H_0(e^{j\omega},\,\mu)| = \left|\cos\left(\tfrac{1}{2}[\Theta_A(\omega,\,\mu) - (N-1)\omega]\right)\right| \tag{4a}$$

and

$$|H_1(e^{j\omega},\,\mu)| = \left|\sin\left(\tfrac{1}{2}[\Theta_A(\omega,\,\mu) - (N-1)\omega]\right)\right|, \tag{4b}$$

respectively. Due to the properties of all-pass filters, the transfer functions $H_0(z,\,\mu)$ and $H_1(z,\,\mu)$ form a doubly complementary pair, satisfying both the all-pass complementary and the power complementary properties, that is,

$$H_0(z,\,\mu) + H_1(z,\,\mu) = z^{-(N-1)} \tag{5a}$$

$$|H_0(e^{j\omega},\,\mu)|^2 + |H_1(e^{j\omega},\,\mu)|^2 = 1. \tag{5b}$$

III. Problem Statement

The goal is to determine the adjustable parameters such that for each value of μ within $-1 \le \mu \le 1$, the magnitude response of $H_0(z,\,\mu)$ $[H_1(z,\,\mu)]$ closely approximates unity [zero] for $\omega \in [0,\,\omega_p + \alpha\mu]$ and zero [unity] for $\omega \in [\omega_s + \alpha\mu,\,\pi]$. Here, ω_p and ω_s are, respectively, the passband and stopband edge frequencies of the variable digital filter for $\mu = 0$, whereas α determines the desired tuning range of the filter. The doubly complementary property of Eq. (5b) guarantees that where one filter has a passband, the second one has a stopband and vice versa. In addition, for practical stopband attenuation (at least 30 dB), the passband ripple becomes very small. Therefore, the optimization of the overall complementary low-pass/high-pass filter pair can concentrate only on the stopband regions of the filters.

We state the following optimization problem:

Optimization problem: Given N, P, ω_p, ω_s, and α find the adjustable parameter vector Φ as given by

$$\Phi = [c_{01}, \ldots, c_{0N}, c_{11}, \ldots, c_{1N}, \ldots, c_{P1}, \ldots, c_{PN}] \tag{6}$$

to minimize the maximum absolute value of the magnitude errors as given by

$$\epsilon = \max\left\{\epsilon_0(\Phi),\,\epsilon_1(\Phi)\right\}, \tag{7a}$$

where

$$\epsilon_k(\Phi) = \max_{-1\le\mu\le 1}\Big[\max_{\omega\in\Omega_s^{(k)}} |H_k(\Phi,\,e^{j\omega},\,\mu)|\Big] \tag{7b}$$

subject to the constraint that the resulting filter is stable for all the values of μ within $-1 \le \mu \le 1$. In the above equation, the stopband regions of $H_k(z,\,\mu)$ for $k = 0, 1$ are $\Omega_s^{(0)} = [\omega_s + \alpha\mu,\,\pi]$ and $\Omega_s^{(1)} = [0,\,\omega_p + \alpha\mu]$, respectively.

IV. Proposed Two-Step Design Scheme

This section describes the proposed two-step technique for optimizing variable recursive digital filters.

A. Optimization Algorithm

In order to solve the optimization problem stated in the previous section we discretize the range $-1 \le \mu \le 1$ into the points $\mu_j \in [-1,\,1]$ for $j = 1, 2, \ldots, J$ and the stopband regions into the frequency points $\omega_i^{(0)} \in [\omega_s + \alpha\mu_j,\,\pi]$ and $\omega_i^{(1)} \in [0,\,\omega_p + \alpha\mu_j]$ for $i = 1, 2, \ldots, I$ and $j = 1, 2, \ldots, J$. In many cases, $I = 100N$ and $J = 20$ are good selections to arrive at a very accurate solution. The resulting discrete minimax problem is to find the adjustable parameter vector Φ to minimize

$$\epsilon = \max\left\{\epsilon_0(\Phi),\,\epsilon_1(\Phi)\right\}, \tag{8a}$$

where

$$\epsilon_k(\Phi) = \max_{\substack{1\le i\le I\\ 1\le j\le J}} |H_k(\Phi,\,e^{j\omega_i^{(k)}},\,\mu_j)| \tag{8b}$$

subject to the condition that the roots of $A(z,\,\mu_j)$ are inside the unit circle for $\mu_j \in [-1,\,1]$ for $j = 1, 2, \ldots, J$.

The above problem can be solved using a constrained nonlinear optimization algorithm. For this purpose, the function `fminimax` from the optimization toolbox provided by MathWorks, Inc. [20] has been used. When using this function, the user has to provide a function that evaluates the objective function, that is, the error function to be minimized at the given frequency points as well as the gradients of the objective function with respect to the adjustable parameters. In addition, a function evaluating the constraints as well as the gradients of these constraints is required.

B. Algorithm for Finding an Initial Filter

The convergence of the above algorithm to the optimal solution implies a rather good initial solution for the adjustable parameters. An initial solution for further optimization can be derived by utilizing the technique proposed in [16]. In this technique, a set of J optimal fixed-coefficient filters corresponding to the above parameters μ_j for $j = 1, 2, \ldots, J$ are designed. Then based on the resulting coefficient sets, a polynomial approximation is derived for each of the coefficients. The design of filters with fixed coefficients can be performed using, e.g., the Remez multiple exchange algorithm [21], eigenfilter approach [22], or second-order cone programming [23]. Recently, also direct methods have been advanced for designing variable fractional delay all-pass filters [6], [7].

C. Practical Considerations

For this filter structure, there is no simple way to estimate the minimum values of N and P to meet the given specifications. These values can be determined by starting from small values of N and P and then gradually increasing these values and re-designing the filter until the specifications are met.

In order to guarantee that the resulting filter is stable, it is required that the poles of the transfer function are inside the unit circle for all μ_j's. The transfer function under consideration cannot be parameterized in such a form for which the stability can be easily verified, e.g., as a cascade of first- and second-order sections. Therefore, we have used the Schür-Cohn stability test [24]. This test gives for a Nth-order denominator polynomial the stability test parameters k_i

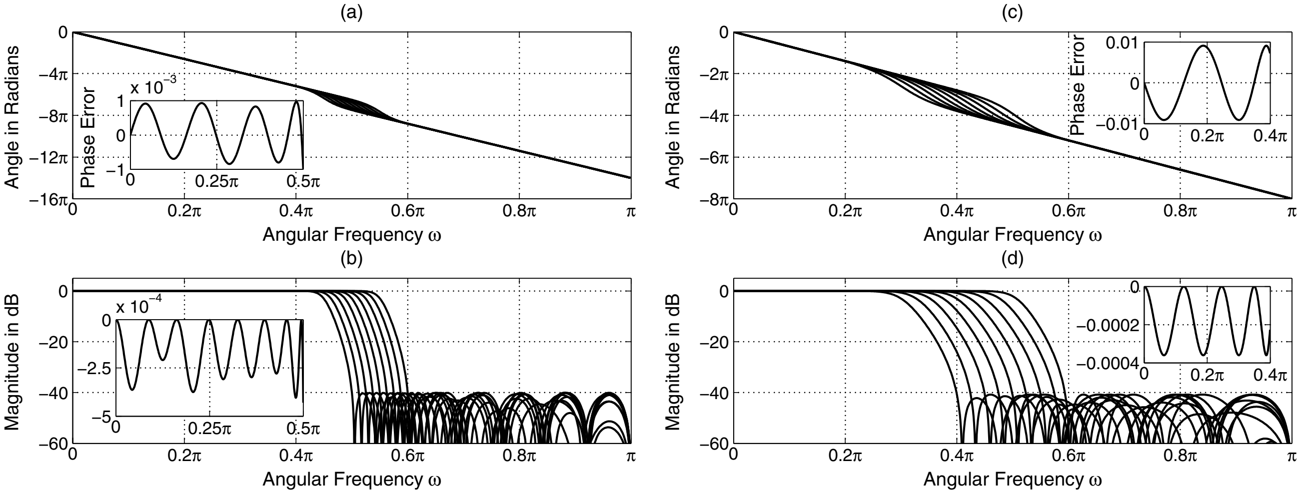

Fig. 3. The phase responses of the optimized variable fractional delay all-pass filter and the magnitude responses for the low-pass part of overall approximately linear-phase variable recursive filter in Examples 1 and 2 for $\mu = -1, -0.75, -0.5, \ldots, 1$. (a) Phase responses and passband phase error for $\mu = 1$ in Example 1. (b) Magnitude responses and passband details for $\mu = 1$ in Example 1. (c) Phase responses and passband phase error for $\mu = 1$ in Example 2. (d) Magnitude responses and passband details for $\mu = 1$ in Example 2.

for $i = 1, 2, \ldots, N-1$. For the stability, it is required that the maximum absolute value of the k_i's is smaller than unity.

The number of adders and multipliers required to implement the overall filter shown in Fig. 2 are $(P+2)N$ and $(P+1)N$, respectively.[1] However, it has turned out that if the resulting filter for $\mu = 0$ is a half-band filter, that is, $\omega_p = \pi - \omega_s$, then the number of adders and multiplier required to implement the overall filter reduces to $(P+2)N/2$ and $(P+1)N/2$, respectively, as will be seen in connection with Example 1.

V. Numerical Examples

This section illustrates, by means of examples, the flexibility and effectiveness of the proposed optimization scheme. In addition, the performance and the complexity of the proposed variable recursive filters are compared with those of the other existing variable recursive filters proposed in the literature.

A. Example 1

It is required that $\omega_p = 0.45\pi + \alpha\mu$, where $\alpha = 0.05\pi$, $\omega_s = \omega_p + 0.1\pi$, and $\delta_s \leq 0.01$. The given specifications are met by $N = 14$ and $P = 4$, that is, the number of multipliers required to implement all the filter coefficients is 70. For the optimized filter, the passband ripple and the stopband attenuation are $4.3 \cdot 10^{-4}$ dB and 40.02 dB, respectively. The maximum phase error is 0.009 98 radians, whereas the radius of the outermost pole is 0.921 96.

The optimized coefficients c_{pn} for $n = 1, 2, \ldots, 14$ and $p = 0, 1, \ldots, 4$ are shown in Table I. As can be seen from this table, the c_{pn}'s are zero-valued for p even and n odd as well as for p odd and n even. Therefore, the number of multipliers required to implement all the filter coefficients reduces to 35. The phase responses for the optimized variable fractional delay all-pass filter $A(z, \mu)$ and the magnitude responses for the low-pass part of the overall approximately linear-phase variable recursive filter pair are depicted in Figs. 3(a) and 3(b) for $\mu = -1, -0.75, -0.5, \ldots, 1$. In addition, the phase error in the passband and the passband details of the magnitude response are shown in these figures in the $\mu = 1$ case.

In this and in the following examples, the optimization has been performed with $I = 100N$ and $J = 20$, whereas the resulting passband ripple and the stopband attenuation as well as the other figures of merit are evaluated with $I = 2^{15}$ and $J = 50$.

B. Example 2

In [10], it is required that $\omega_p = 0.3\pi + \alpha\mu$, where $\alpha = 0.1\pi$, $\omega_s = \omega_p + 0.2\pi$, and $\delta_s \leq 0.01$ The passband ripple and the stopband attenuation for the variable recursive digital filter in [10] are 0.062 dB and 38.29 dB, respectively, whereas the total number of multipliers needed to implement the overall filter is 117.[2]

For the proposed structure, these specifications are met by $N = 8$ and $P = 3$, that is, the overall number coefficients is only 32. The passband ripple and the stopband attenuation for the optimized filter are $4.2 \cdot 10^{-4}$ dB and 40.35 dB, respectively, whereas the maximum phase error is 0.009 81 radians. The optimized coefficients c_{pn} for $n = 1, 2, \ldots, 8$ and $p = 0, 1, 2, 3$ are shown in Table II. The phase and magnitude responses for the optimized filters are depicted in Figs. 3(c) and 3(d). In addition, the passband details of the phase error and the magnitude response are shown in these figures in the $\mu = 1$ case.

C. Example 3

In [9], [16], the desired magnitude characteristics of a variable low-pass filter are stated as follows: unity for $\omega \in [0,\, 0.26\pi + \Psi]$ and zero for $\omega \in [0.5\pi + \Psi,\, \pi]$. Here, the parameter Ψ is in the range $[-0.16\pi,\, 0.16\pi]$. In [9], the numerator for a recursive filter used to meet the specifications is of order four and the denominator is a cascade of two second-order sections, whereas the degree of the polynomial approximation is four ($P = 4$), that is, the overall number of coefficients is 45. For the optimized filter in [9], the passband ripple and the stopband attenuation are $7.24 \cdot 10^{-2}$ dB and 23.65 dB, respectively.

For the proposed filter structure, only 12 coefficients are needed ($N = 4$ and $P = 2$) to meet approximately the same specifications. The passband ripple and the stopband attenuation are $1.0 \cdot 10^{-2}$ dB and 26.52 dB, respectively, whereas the maximum phase error is 0.0480 radians.

[1] The number of multipliers required to implement the multiplications by μ^p for $p = 1, 2, \ldots, P$ are not included in this figure.

[2] It should be pointed out that the filter structure proposed in [10] is free from transients at the filter output during the changes in the filter parameters.

TABLE I

OPTIMIZED COEFFICIENT VALUES FOR THE VARIABLE RECURSIVE DIGITAL FILTER IN EXAMPLE 1 ($N = 14$ AND $P = 4$)

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$	$n=12$	$n=13$	$n=14$
c_{0n}	0	0.4970	0	−0.1168	0	0.0560	0	−0.0304	0	0.0184	0	−0.0136	0	0.0086
c_{1n}	0.1542	0	−0.0760	0	0.0527	0	−0.0404	0	0.0303	0	−0.0199	0	0.0334	0
c_{2n}	0	−0.0157	0	0.0104	0	−0.0232	0	0.0200	0	−0.0214	0	0.0231	0	−0.0143
c_{3n}	0.0030	0	0.0016	0	−0.0026	0	0.0064	0	−0.0078	0	0.0078	0	−0.0176	0
c_{4n}	0	0.0028	0	0.0055	0	0.0035	0	−0.0013	0	0.0049	0	−0.0049	0	0.0018

TABLE II

OPTIMIZED COEFFICIENT VALUES FOR THE VARIABLE RECURSIVE DIGITAL FILTER IN EXAMPLE 2 ($N = 8$ AND $P = 3$)

$c_{01} = -0.3078$	$c_{02} = 0.4387$	$c_{03} = 0.1271$	$c_{04} = -0.0492$
$c_{11} = 0.2840$	$c_{12} = 0.0833$	$c_{13} = -0.0929$	$c_{14} = -0.0891$
$c_{21} = 0.0265$	$c_{22} = -0.0349$	$c_{23} = -0.0453$	$c_{24} = 0.0199$
$c_{31} = 0.0174$	$c_{32} = 0.0010$	$c_{33} = 0.0006$	$c_{34} = 0.0130$
$c_{05} = -0.0568$	$c_{06} = -0.0065$	$c_{07} = 0.0241$	$c_{08} = 0.0061$
$c_{15} = 0.0136$	$c_{16} = 0.0608$	$c_{17} = 0.0258$	$c_{18} = -0.0201$
$c_{25} = 0.0480$	$c_{26} = 0.0122$	$c_{27} = -0.0379$	$c_{28} = -0.0132$
$c_{35} = -0.0009$	$c_{36} = -0.0234$	$c_{37} = -0.0184$	$c_{38} = 0.0055$

TABLE III

SUMMARY OF FILTER DESIGNS IN EXAMPLES 1, 2, AND 3.

		A_p (dB)	A_s (dB)	Δ_p (rad)	$r_{\max}$	$N_{\rm M}$	$N_{\rm O}$
Ex. 1	Proposed	$4.33 \cdot 10^{-4}$	40.02	$9.98 \cdot 10^{-3}$	0.922	35	82
Ex. 2	In [10]	$6.2 \cdot 10^{-2}$	38.29	$\approx 0.08^a$	–	117	244
	Proposed	$4.18 \cdot 10^{-4}$	40.35	$9.81 \cdot 10^{-3}$	0.869	32	76
Ex. 3	In [9]	$7.24 \cdot 10^{-2}$	23.65	$1.62 \cdot 10^{-1}$	0.825	45	99
	Proposed	$9.99 \cdot 10^{-3}$	26.52	$4.80 \cdot 10^{-2}$	0.807	12	31

[a]Here, Δ_p denotes the group delay error for the optimized filter in [10].

D. Summary

The summary for the filters optimized in Examples 1, 2, and 3 is shown in Table III. In this table, A_p, A_s, and Δ_p denote the passband ripple in decibels, the stopband attenuation in decibels, and the maximum phase error in radians, respectively, whereas $r_{\max}$, $N_{\rm M}$, and $N_{\rm O}$ denote, respectively, the radius of outermost the pole, the number of multipliers required to implement all the filter coefficients, and the number of arithmetic operations per sample required for implementing the overall filter pair.

VI. CONCLUSION

This paper proposes a new structure for implementing complementary low-pass/high-pass approximately linear-phase variable recursive digital filters. It has been shown that significant savings in the implementation cost are achieved by using the proposed structure. In addition, a two-step algorithm has been proposed for optimizing these filters.

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