Design of 1-D Stable Variable Fractional Delay IIR Filters

Hui Zhao and Hon Keung Kwan

***Abstract*—In this brief, a two-stage approach for the design of 1-D stable variable fractional delay infinite-impulse response (IIR) digital filters is proposed. In the first stage, a set of fixed delay stable IIR filters are designed by minimizing a quadratic objective function, which is defined by integrating error criterion with IIR filter stability constraint condition. Then, the final design is determined by fitting each of the fixed delay filter coefficients as a 1-D polynomial. Two design examples are given to show the effectiveness of the proposed design method.**

***Index Terms*—Digital filters, infinite-impulse response (IIR) filters, variable fractional delay (VFD) filters.**

I. Introduction

VARIABLE fractional delay (VFD) digital filter design has received increasing attention in recent years. Although advances have been made on the design of finite-impulse response (FIR) and allpass types of VFD filters, for examples [1]–[8], it will be of interest to investigate approaches for the design of VFD infinite-impulse response (IIR) filters, since this type of filters can improve computation efficiency and reduce system delay compared to what can be achieved by FIR filters. For the purpose of reducing the computational complexity for the design of variable digital filters, a two-stage approach, i.e., designing a set of fixed-coefficient filters, and then fitting each of the coefficients by a polynomial function, has been proposed in the literatures [9], [10]. In [11], we have presented an efficient two-stage approach for the design of 1-D VFD FIR filters.

In this brief, a two-stage approach for the design of 1-D VFD IIR filters is presented. In Section II, we introduce the design problem. Then in Section III, the design method is formulated and the computational complexity is evaluated. In Section IV, two design examples, together with their comparisons with other 1-D VFD allpass filters, are given to illustrate the effectiveness of the proposed design method.

II. Design Problem

Let the frequency response of the desired 1-D VFD filter be

$$H(j\omega, p) = \mathrm{e}^{-j\omega(D+p)}, \qquad \omega \in [0, \omega_c] \tag{1}$$

Manuscript received August 1, 2005; revised January 25, 2006 and May 9, 2006. This paper was recommended by Associate Editor S. L. Netto.

H. Zhao is with School of Automation Engineering, University of Electronic Science and Technology of China, Chengdu, Sichuan 610054, China (e-mail: zhaohui@uestc.edu.cn).

H. K. Kwan is with Department of Electrical and Computer Engineering, University of Windsor, ON N9B 3P4, Canada (e-mail: kwan1@uwindsor.ca).

Digital Object Identifier 10.1109/TCSII.2006.882802

where $D \geq 0$ is the mean delay, p is the adjustable parameter representing the variable fractional delay on $[-0.5, 0.5]$, and ω_c is the filter cutoff frequency. The design task is to find a stable and variable IIR filter with finite coefficients $a_i(p)(i = 1, 2, \cdots P)$ and $b_i(p)(i = 0, 1, \cdots Q)$, whose frequency response

$$H_F(j\omega, p) = \frac{B(j\omega, p)}{A(j\omega, p)} = \frac{\sum\limits_{i=0}^{Q} b_i(p)\mathrm{e}^{-ji\omega}}{1 + \sum\limits_{i=1}^{P} a_i(p)\mathrm{e}^{-ji\omega}} \tag{2}$$

approximates to $H(j\omega, p)$. Moreover, it is well known that to guarantee the filter designed to be stable, constraint must be imposed on the coefficients $a_i(p)$. The sufficient stability condition is given in [12] as

$$\mathrm{R_e}\left[A(j\omega, p)\right] = 1 + \sum_{i=1}^{P} a_i(p)\cos(i\omega) > 0, \quad \omega \in [0, \pi). \tag{3}$$

For the variable coefficients $a_i(p)$ and $b_i(p)$ being continuous functions of p, we assume, like those in other papers, that $a_i(p)$ and $b_i(p)$ are polynomial functions of p with the degree K, i.e.

$$a_i(p) = \sum_{k=0}^{K} x(i,k)p^k \tag{4a}$$

$$b_i(p) = \sum_{k=0}^{K} y(i,k)p^k. \tag{4b}$$

Therefore, the design task can be expressed as the following constrained optimization problem.

Find coefficients $x(i,k)(i = 1, 2, \cdots P, k = 0, 1, \cdots K)$ and $y(i,k)(i = 0, 1, \cdots Q, k = 0, 1, \cdots K)$ while $H_F(j\omega, p)$ approximates to $H(j\omega, p)$, subject to $\mathrm{R_e}[A(j\omega, p)] > 0$, for all $\omega \in [0, \pi)$ and $p \in [-0.5, 0.5]$.

III. Design Method

In this brief, a two-stage approach is employed to solve the 1-D VFD IIR filter design problem.

A. First Stage

First of all, we sample uniformly $M + 1$ fractional delay points $p_m = (m/M) - 0.5(m = 0, 1, \cdots M)$ on $[-0.5, 0.5]$ to form $M + 1$ ideal fixed fractional delay filters with frequency response $H(j\omega,p_m)$ for $m=0,1,\cdots M$. We define $M+1$ error functions as

$$e_1^2(p_m)=\int_0^{\omega_c}|B(j\omega,p_m)-H(j\omega,p_m)A(j\omega,p_m)|^2\,d\omega. \quad (5)$$

When $e_1^2(p_m)$ reaches its global minimum, the designed coefficients $a_i(p_m)(i=1,2,\cdots P)$ and $b_i(p_m)(i=0,1,\cdots Q)$ of the fixed delay IIR filter, such that the corresponding $H_F(j\omega,p_m)$ approximates to $H(j\omega,p_m)$, can be obtained. Furthermore, from (3), we have

$$1+\sum_{i=1}^{P}a_i(p_m)\cos(i\omega)\geq 1-\sum_{i=1}^{P}|a_i(p_m)|\geq 1-\sqrt{2\sum_{i=1}^{P}a_i^2(p_m)}. \quad (6)$$

We now define a quadratic function for the stability constraint as

$$e_2^2(p_m)=\sum_{i=1}^{P}a_i^2(p_m) \quad (7)$$

It can be shown that if $e_2^2(p_m)$ is less than 0.5, the stability constraint condition (3) is satisfied for all $\omega\in[0,\pi)$. Now, by combining (5) and (7), we define the objective function as

$$e^2(p_m)=e_1^2(p_m)+W_s e_2^2(p_m) \quad (8)$$

where $W_s>0$ is an adjustable positive number and shall be called stability control parameter in this brief. Therefore, the task to design a stable fixed delay IIR filter is to minimize $e^2(p_m)$. It is observed that to facilitate the design of a stable filter, W_s must be taken big enough during minimizing $e^2(p_m)$. From (8), it is clear that the objective function $e^2(p_m)$ will go to its minimum when

$$\frac{\partial e^2(p_m)}{\partial a_i(p_m)}=0, \qquad i=1,2,\cdots P \quad (9a)$$

$$\frac{\partial e^2(p_m)}{\partial b_i(p_m)}=0, \qquad i=0,1,\cdots Q. \quad (9b)$$

Since the cutoff frequency ω_c is always taken as $\omega_c<\pi$ in practical designing process, it can be proved that (9) have such a unique solution

$$\begin{aligned}\boldsymbol{A}(p_m)=&\left[(\boldsymbol{S}_2+\boldsymbol{W}_s\boldsymbol{I}_P)^{-1}\boldsymbol{S}_1(p_m)\boldsymbol{S}_3^{-1}\boldsymbol{S}_1(p_m)^T-\boldsymbol{I}_P\right]^{-1}\\&\times(\boldsymbol{S}_2+\boldsymbol{W}_s\boldsymbol{I}_P)^{-1}\\&\times\left[\boldsymbol{S}_1(p_m)\boldsymbol{S}_3^{-1}\boldsymbol{R}_b(p_m)-\boldsymbol{R}_a\right] \quad (10a)\end{aligned}$$

$$\begin{aligned}\boldsymbol{B}(p_m)=&\left[\boldsymbol{S}_3^{-1}\boldsymbol{S}_1(p_m)^T(\boldsymbol{S}_2+\boldsymbol{W}_s\boldsymbol{I}_P)^{-1}\boldsymbol{S}_1(p_m)-\boldsymbol{I}_{Q+1}\right]^{-1}\\&\times\boldsymbol{S}_3^{-1}\left[\boldsymbol{S}_1(p_m)^T(\boldsymbol{S}_2+\boldsymbol{W}_s\boldsymbol{I}_P)^{-1}\boldsymbol{R}_a\right.\\&\left.-\boldsymbol{R}_b(p_m)\right] \quad (10b)\end{aligned}$$

where

$$\boldsymbol{A}(p_m)=[a_1(p_m)\quad a_2(p_m)\quad\cdots\quad a_P(p_m)]^T \quad (11a)$$

$$\boldsymbol{B}(p_m)=[b_0(p_m)\quad b_1(p_m)\quad\cdots\quad b_Q(p_m)]^T \quad (11b)$$

In (10), $\boldsymbol{I}_P$ and $\boldsymbol{I}_{Q+1}$ are, respectively, $P\times P$ and $(Q+1)\times(Q+1)$ unit matrices. The matrices or vectors $\boldsymbol{S}_1(p_m)$, $\boldsymbol{S}_2$, $\boldsymbol{S}_3$, $\boldsymbol{R}_a$ and $\boldsymbol{R}_b(p_m)$ are, respectively, of dimensions $P\times(Q+1)$, $P\times P$, $(Q+1)\times(Q+1)$, $P\times 1$ and $(Q+1)\times 1$, with their elements defined by

$$s_1(j,i)(p_m)=-\int_0^{\omega_c}\cos(D+p_m+j-i+1)\omega d\omega,$$
$$j=1,2,\cdots P,\ \ i=1,2,\cdots Q+1 \quad (12a)$$

$$s_2(j,i)=\int_0^{\omega_c}\cos(j-i)\omega d\omega,\ \ j,i=1,2,\cdots P \quad (12b)$$

$$s_3(j,i)=\int_0^{\omega_c}\cos(j-i)\omega d\omega,\ \ j,i=1,2,\cdots Q+1 \quad (12c)$$

$$r_a(j,1)=-s_2(j,0),\ \ j=1,2,\cdots P \quad (12d)$$

$$r_b(j,1)(p_m)=-s_1(0,j)(p_m),\ \ j=1,2,\cdots Q+1. \quad (12e)$$

Solving (10) for $m=0,1,\cdots M$, the $M+1$ coefficients $\boldsymbol{A}(p_m)$ and $\boldsymbol{B}(p_m)$ such that $H_F(j\omega,p_m)$ approximating $H(j\omega,p_m)$ can be obtained. If the stability control parameter W_s is taken big enough, the corresponding $M+1$ fixed delay IIR filters obtained are usually stable.

B. Second Stage

Based on the above obtained coefficients $\boldsymbol{A}(p_m)$ and $\boldsymbol{B}(p_m)(m=0,1,\cdots M)$, the $P+Q+1$ polynomial fitting error functions are defined by

$$e_a^2(i)=\sum_{m=0}^{M}\left[\sum_{k=0}^{K}x(i,k)p_m^k-a_i(p_m)\right]^2,$$
$$i=1,2,\cdots P \quad (13a)$$

$$e_b^2(i)=\sum_{m=0}^{M}\left[\sum_{k=0}^{K}y(i,k)p_m^k-b_i(p_m)\right]^2,$$
$$i=0,1,\cdots Q. \quad (13b)$$

By minimizing $e_a^2(i)$ and $e_b^2(i)$, we can obtain the optimal fitting coefficients $x(i,k)(k=0,1,\cdots K)$ and $y(i,k)(k=0,1,\cdots K)$. To solve this problem, we make $\partial e_a^2(i)/\partial x(i,k)=0$ and $\partial e_b^2(i)/\partial y(i,k)=0$ for $k=0,1,\cdots K$, and then arrive at the following unique solutions:

$$\boldsymbol{X}(i)=\boldsymbol{C}^{-1}\boldsymbol{D}(i) \quad (14a)$$

$$\boldsymbol{Y}(i)=\boldsymbol{C}^{-1}\boldsymbol{E}(i) \quad (14b)$$

where

$$\boldsymbol{X}(i)=[x(i,0)\quad x(i,1)\quad\cdots\quad x(i,K)]^T \quad (15a)$$

$$\boldsymbol{Y}(i)=[y(i,0)\quad y(i,1)\quad\cdots\quad y(i,K)]^T. \quad (15b)$$

$\boldsymbol{C}, \boldsymbol{D}(i)$ and $\boldsymbol{E}(i)$, are respectively, $(K+1)\times(K+1)$, $(K+1)\times 1$ and $(K+1)\times 1$ matrices or vectors which can be expressed as

$$c_{kl} = \sum_{m=0}^{M}\left(\frac{m}{M}\right)^{k+l-2}, \qquad k,l=1,2,\cdots K+1 \tag{16a}$$

$$d_k(i) = \sum_{m=0}^{M}\left(\frac{m}{M}\right)^{k-1} a_i(p_m), \qquad k=1,2,\cdots K+1 \tag{16b}$$

$$e_k(i) = \sum_{m=0}^{M}\left(\frac{m}{M}\right)^{k-1} b_i(p_m), \qquad k=1,2,\cdots K+1. \tag{16c}$$

Finally, by solving (14a) for $i = 1,2,\cdots P$ and (14b) for $i = 0,1,\cdots Q$, optimal fitting coefficients $\boldsymbol{X}(i)(i=1,2,\cdots P)$ and $\boldsymbol{Y}(i)(i=0,1,\cdots Q)$ can be obtained.

C. Remarks

Based on the above formulation the following are true.

1) The design of 1-D stable VFD IIR filters is performed by solving in tandem the matrix equations (10) and (14). And, the filter obtained is stable, if the stability control parameter W_s is taken big enough in the design.
2) From (1) and (4), the ideal frequency responses $H(j\omega,p)$, the filter coefficients $a_i(p)$ and $b_i(p)$ are all continuous functions of the fractional delay p. So theoretically speaking, $H_F(j\omega,p)$ can fit the ideal response $H(j\omega,p)$ well, provided the sampling scale M is taken large enough and the polynomial degree K is also taken high enough.
3) From (10) and (14), the major computation involves calculating the inverse matrices $[\boldsymbol{S}_3^{-1}\boldsymbol{S}_1(p_m)^T(\boldsymbol{S}_2+\boldsymbol{W}_s\boldsymbol{I}_P)^{-1}\boldsymbol{S}_1(p_m)-\boldsymbol{I}_{Q+1}]^{-1}$, $[(\boldsymbol{S}_2+\boldsymbol{W}_s\boldsymbol{I}_P)^{-1}\boldsymbol{S}_1(p_m)\boldsymbol{S}_3^{-1}\boldsymbol{S}_1(p_m)^T-\boldsymbol{I}_P]^{-1}$ and $\boldsymbol{C}^{-1}$, with matrix dimensions, respectively, $P\times P$, $(Q+1)\times(Q+1)$ and $(K+1)\times(K+1)$. It is known that an $n\times n$ inverse matrix can be obtained using n^3 floating-point operations (flops). Therefore, the computational complexity of the proposed method is $(M+2)(Q+1)^3+(M+2)P^3+(K+1)^3$ flops.
4) It is known that the error function $e_1^2(p_m)$ is not true mean-squared error (MSE) function, which should be

$$e_{\mathrm{MSE}}^2(p_m) = \int_0^{\omega_c}\frac{|B(j\omega,p_m)-H(j\omega,p_m)A(j\omega,p_m)|^2}{|A(j\omega,p_m)|^2}d\omega. \tag{17}$$

To reduce the design complexity, we ignore the denominator $|A(j\omega,p_m)|^2$. In fact, this simplification yields group delay response that fits the desired ones well enough. Although it can not strictly control the magnitude response, but as shown in the following design examples, it can still obtain satisfactory design results.

IV. Design Examples

In this section, we will present two examples simulated using MATLAB on a PC to show the effectiveness of the proposed approach for designing 1-D stable VFD IIR filters.

To evaluate the performance, the frequency response error $e_H(\omega,p)$, maximum frequency response error e_{Max}, normalized root mean square (rms) error e_{rms}, fractional group delay error $e_p(\omega,p)$ and maximum fractional delay error $e_{p\mathrm{Max}}$ are defined, as those adopted in [7], by

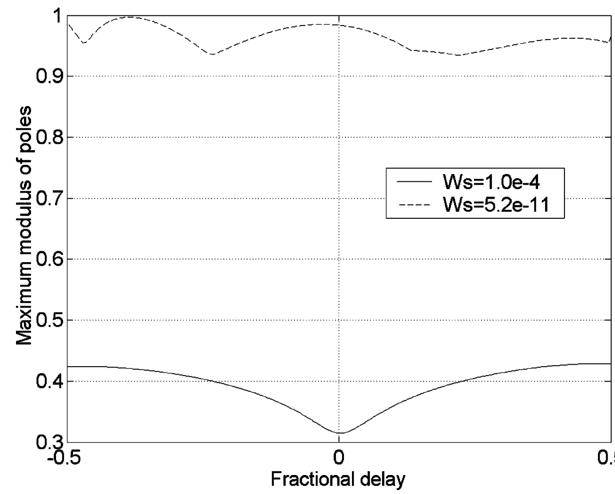

Fig. 1. Maximum modulus of filter poles of Example 1.

$$e_H(\omega,p) = |H_F(j\omega,p)-H(j\omega,p)| \tag{18}$$

$$e_{\mathrm{MAX}} = \max\{e_H(\omega,p), \omega\in[0,0.9\pi], p\in[-0.5,0.5]\} \tag{19}$$

$$e_{\mathrm{rms}} = \left[\frac{\int_0^{0.9\pi}\int_{-0.5}^{0.5}|e_H(\omega,p)|^2\,dpd\omega}{\int_0^{0.9\pi}\int_{-0.5}^{0.5}|H(j\omega,p)|^2\,dpd\omega}\right]^{0.5} \tag{20}$$

$$e_p(\omega,p) = |\tau(\omega,p)-p| \tag{21}$$

$$e_{p\mathrm{MAX}} = \max\{e_p(\omega,p), \omega\in[0,0.9\pi], p\in[-0.5,0.5]\}. \tag{22}$$

In (21), $\tau(\omega,p)$ and p are the actual and desired fractional group delay responses.

Example 1: In [5]–[8], the filter numerator and denominator polynomials must be of equal order for the design of fractional delay allpass filters. In this brief, no such constraint is required. The design specifications are chosen as $\omega_c = 0.9\pi$, $Q = 55$, $P = 14$, $D = 27$, $K = 5$, $M = 11$. From Section III, it is known that the stability control parameter W_s has to be chosen carefully. It is found in our design process that if W_s is larger than about 5.2×10^{-11}, the filter obtained will be stable. As shown in Fig. 1 (dashed), when $W_s = 5.2\times10^{-11}$, the maximum modulus of filter poles for $p\in[-0.5,0.5]$ is 0.9967, hence the VFD IIR filter obtained is stable. Moreover, different W_s will yield different filter performance. Fig. 2 shows e_{rms} decreases as W_s increases from 1.0×10^{-11} to 1.0×10^{-3}. It can be seen that e_{rms} will reach its minimum if W_s is bigger than about 1.0×10^{-6}, i.e., e_{rms} does not change when W_s increases beyond 1.0×10^{-6}.

Now, we show the performances of the 1-D stable VFD IIR filter designed by the proposed method, when the stability control parameter is taken as $W_s = 1.0\times10^{-4}$. Figs. 3–5 show, respectively, the frequency response error $e_H(\omega,p)$, fractional group delay response $\tau(\omega,p)$ and fractional group delay error

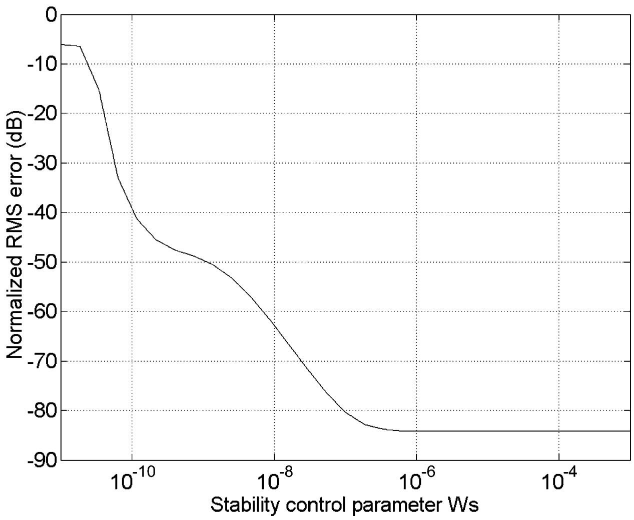

Fig. 2. Different W_s yields different $e_{\rm rms}$ in Example 1.

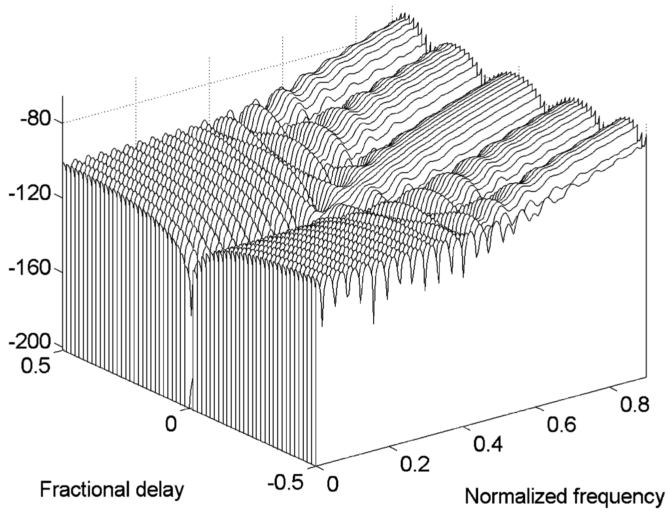

Fig. 3. Frequency response error in decibels of Example 1.

$e_p(\omega, p)$ at the frequency range $[0, 0.9\pi]$. The maximum modulus of filter poles for $p \in [-0.5, 0.5]$ is also shown in Fig. 1 (solid line). Because the maximum is 0.4289, the designed filter is stable.

From Table I, the normalized rms error $e_{\rm rms}$ is larger than those of [6], [7], but the maximum fractional delay error $e_{p\rm Max}$ is smaller than that of [6]. Although the designed filter has $70 \times 6 = 420$ coefficients which is two times of those of the two allpass filters of [6], [7], it achieves the lowest mean delay.

Example 2: Adopting the same order of 35 as the allpass filters [6], [7], we select the design specifications as $\omega_c = 0.9\pi$, $Q = 35$, $P = 35$, $D = 35$, $K = 5$, $M = 11$. It is found in our design process that even when the stability control parameter W_s is taken zero the filter obtained is still stable. The corresponding magnitude response and fractional group delay response are obtained as shown in Figs. 6 and 7. It can be seen that the filter has produced a better approximation to the allpass character. The maximum modulus of filter poles for $p \in [-0.5, 0.5]$ is 0.9285. Hence, the filter is stable. Therefore, a 1-D allpass-like VFD IIR filter can be designed by using the proposed approach with equal orders of numerator and denominator polynomials.

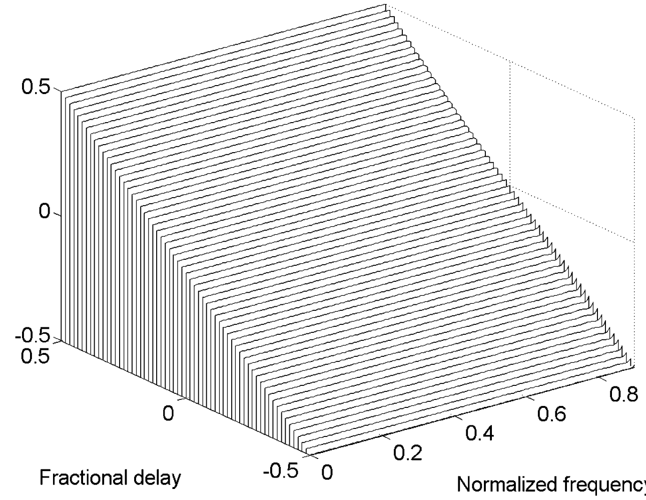

Fig. 4. Fractional group delay response of Example 1.

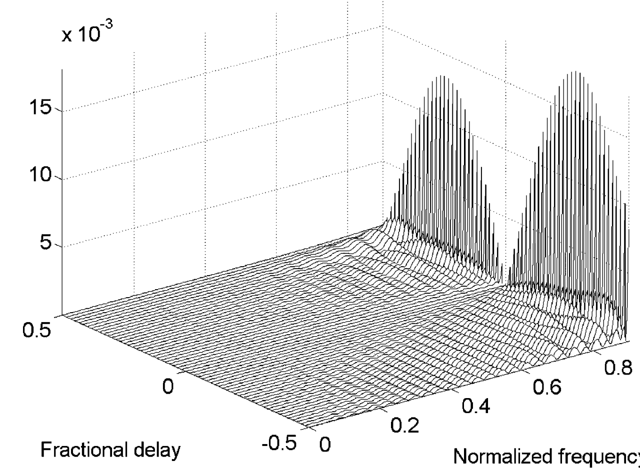

Fig. 5. Fractional group delay error of Example 1.

TABLE I
DESIGN RESULTS COMPARED WITH THOSE OF [6] AND [7]

Method	$e_{\rm rms}$ (%)	$e_{\rm Max}$ (dB)	$e_{p\rm Max}$	Mean delay
Proposed design*	0.006937	-65.51	0.0181	27
Proposed design**	0.000623	-76.49	0.0288	35
LS design of [7]	0.000562	-74.95	0.0158	35
LS design of [6]	0.001146	-65.14	0.0369	35

* Example 1 **Example 2

The performances are also listed in Table I. It can be found that the filter designed, which can approximate well an allpass filter, has performances better than those of [6] and comparable with those of [7].

It should be noted that all of the performance indexes in Table I are computed under the same condition as: 201 discrete normalized frequency points on $\omega \in [0, 0.9\pi]$ and 61 fractional normalized delay points on $p \in [-0.5, 0.5]$. And, to design the above two filters, the proposed approach needs less than 0.4 and 0.5 seconds CPU time, respectively, on a PC (Pentium IV, 1.8 GHz).

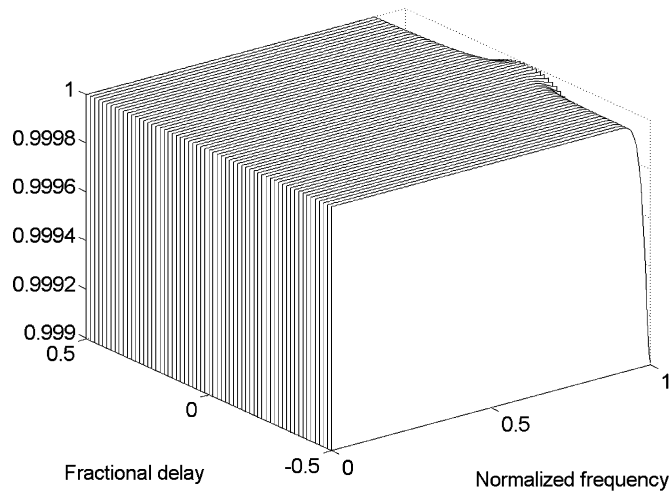

Fig. 6. Magnitude response of Example 2.

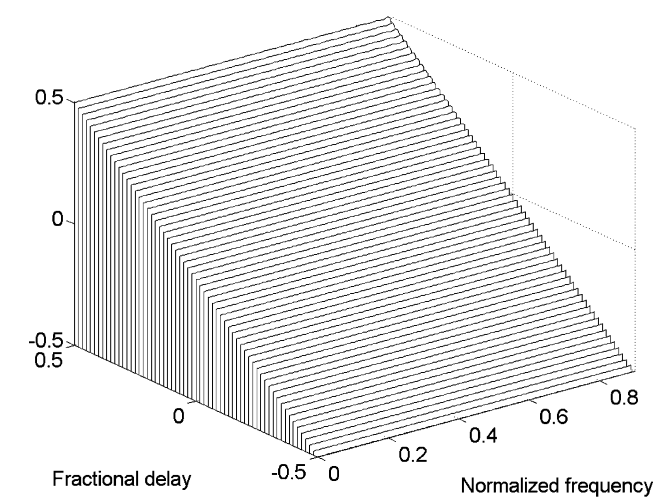

Fig. 7. Fractional group delay response of Example 2.

V. Conclusion

In this brief, a two-stage approach for the design of 1-D stable VFD IIR filters has been proposed. The proposed method employs in tandem two designing steps. Firstly, a set of fixed delay stable IIR filters are designed by minimizing a quadratic objective function formulated in this brief. The design tasks are accomplished by solving a set of linear matrix equations. And then, the final design result is determined by fitting the fixed delay filter coefficients as 1-D polynomials. In the proposed method, the objective function is defined in terms of the approximating error criterion and the IIR filter stability constraint. Therefore, the filter designed is guaranteed to be stable while the computational complexity is reduced. To show its effectiveness, two design examples, together with their comparisons with other design methods for 1-D VFD allpass filters have been presented.

References

[1] W. S. Lu and T. B. Deng, "An improved weighted least-squares design for variable fractional delay FIR filters," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 46, no. 8, pp. 1035–1040, Aug. 1999.

[2] J. Vesma and T. Saramaki, "Design and properties of polynomial-based fractional delay filters," in *Proc. Int. Symp. Circuits Syst.*, 2000, pp. I-104–I-107.

[3] T. B. Deng, "Discretization-free design of variable fractional-delay FIR filters," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 48, no. 6, pp. 637–644, Jun. 2001.

[4] H. Johansson and P. Löwenborg, "On the design of adjustable fractional delay FIR filters," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 50, no. 4, pp. 164–169, Apr. 2003.

[5] S. C. Pei and P. H. Wang, "Closed-form design of all-pass fractional delay filters," *IEEE Trans. Signal Proces. Lett.*, vol. 11, no. 10, pp. 788–791, Oct. 2004.

[6] C. C. Tseng, "Design of 1-D and 2-D variable fractional delay allpass filters using weighted least-squares method," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 49, no. 10, pp. 1413–1422, Oct. 2002.

[7] T. B. Deng, "Noniterative WLS design of allpass variable fractional-delay digital filters," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 53, pp. 358–371, Feb. 2006.

[8] J, Yli-Kaakinen and T. Saramaki, "An algorithm for the optimization of adjustable fractional-delay all-pass filters," in *Proc. Int. Symp. Circuits Syst.*, 2004, vol. 3, pp. 153–156.

[9] R. Zarour and M. M. Fahmy, "A design technique for variable digital filters," *IEEE Trans. Circuits Syst.*, vol. 36, no. 11, pp. 1473–1478, Nov. 1989.

[10] T. B. Deng, "Design of recursive 1-D variable filters with guaranteed stability," *IEEE Trans. Circuits Syst. II, Analog Digit. Signal Process.*, vol. 44, no. 9, pp. 689–695, Sep. 1997.

[11] H. Zhao and J. Yu, "A simple and efficient design of variable fractional delay FIR Filters," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, vol. 53, pp. 157–160, Feb. 2006.

[12] A. T. Chottera and G. A. Jullien, "A linear programming approach to recursive filter design with linear phase," *IEEE Trans. Circuits Syst.*, vol. CAS-29, no. 3, pp. 139–149, Mar. 1982.