MILP DESIGN OF FREQUENCY-RESPONSE MASKING FIR FILTERS WITH FEW SPT TERMS

Oscar Gustafsson, Håkan Johansson, and Lars Wanhammar

Department of Electrical Engineering, Linköping University, SE-581 83 Linköping, SWEDEN
Tel: +46-13-28 {4059, 1676, 1344}, Fax: +46-13-13 92 82, E-mail: {oscarg, hakanj, larsw}@isy.liu.se

ABSTRACT

In this work we formulate a mixed integer linear programming (MILP) problem that minimizes the number of signed-power-of-two (SPT) terms given a filter specification for linear-phase frequency-response masking (FRM) filters. The proposed method designs the filters in two steps. The model filter and the masking filters are designed separately, but subject to each other. Hence, it is not guaranteed that the global minimum is obtained. However, each solution will be optimal given the other filter(s), and iteration may improve the overall solution. The filter design problems are formulated using normalized peak ripple magnitude (NPRM), which for FRM filters introduces some implications, which is also discussed in this work.

1. INTRODUCTION

The problem of finding fixed-point multiplier coefficients for digital filters has been extensively studied over the years [1]. A large part of the work has considered finding filter coefficients with few signed-power-of-two (SPT) terms. This is motivated by that the number of additions required to realize a multiplication is the number of SPT terms minus one.

On the other hand, work has been done that consider the number of multiplications in an FIR filter. By introducing new filter structures, the number of arithmetic operations has been reduced [2].

In this work we present an approach to combine some of these two approaches. We formulate a mixed integer linear programming (MILP) problem for frequency-response masking (FRM) filters [3]. These types of filters have been shown to reduce the arithmetic complexity significantly for linear-phase FIR filters with narrow transition band [3]. Previous approaches minimized the ripple given a hardware specification [4]–[6]. In this work, we minimize the hardware cost (number of SPT terms) given a filter specification.

2. FREQUENCY-RESPONSE MASKING FILTERS

A linear-phase arbitrary-bandwidth FIR frequency-response masking (FRM) filter consists of a periodic *model filter*, its complement and two *masking filters*, both with the same delay, as shown in Fig. 1. For an explanation of the principles we refer to [3]. Figure 1 also illustrates the notation used throughout this work.

For a frequency-response masking filter, the overall transfer function can be expressed as

$$H(z) = G(z^L)E(z) + G_c(z^L)F(z) \quad (1)$$

where $G_c(z)$ denotes the complementary filter of $G(z)$, given by

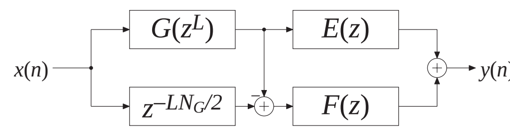

Figure 1. Frequency-response masking filter structure.

$$G_c(z) = z^{-N_G/2} - G(z) \quad (2)$$

with N_G being the order of $G(z)$.

3. LINEAR-PHASE FIR FILTERS

The frequency response of a linear-phase FIR filter can be separated into a real-valued function $H_R(\omega T)$ and a real-valued phase function $\Theta(\omega T)$ as

$$H(e^{j\omega T}) = H_R(\omega T)e^{j\Theta(\omega T)} \quad (3)$$

where $H_R(\omega T)$ is the *zero-phase frequency response*. We have $|H(e^{j\omega T})| = |H_R(\omega T)|$, as $|e^{jx}| = 1$ for real-valued x.

For a linear-phase FIR filter with filter order N and symmetric impulse response, the zero-phase frequency response can be written as

$$H_R(\omega T) = \sum_{m=1}^{M} h_m c(m, \omega T) \quad (4)$$

where M is the number of filter coefficients utilizing symmetry and $c(m,\omega T)$ is a trigonometric function depending on if the filter order is even or odd [2].

Denote the desired zero-phase frequency response by $D(\omega T)$. $D(\omega T)$ is normally 1 in the passband and 0 in the stopband. If the maximal allowed deviation is denoted by $\delta(\omega T)$ the constraints on the filter can be written as

$$|H_R(\omega T) - D(\omega T)| \le \delta(\omega T) \quad (5)$$

or, equivalently,

$$-\delta(\omega T) \le H_R(\omega T) - D(\omega T) \le \delta(\omega T) \quad (6)$$

For a given angle, ωT, (6) is an inequality that is linear in the filter coefficients, h_n. Thus, by expressing (6) for a grid of different angles, ωT, a set of inequalities are obtained that can be used for the filter design. However, even if a solution is found that is valid for all angles on the grid it can still violate the original specification for an angle that is not on the grid. Hence, the obtained solution must be checked on a much denser grid.

Note that there are no restrictions on the functions $D(\omega T)$ and $\delta(\omega T)$. It is therefore possible to design any type of linear-phase FIR filters as long as the specifications can be expressed using the zero-phase frequency function.

4. PROPOSED DESIGN METHOD

4.1. Signed-Powers-of-Two Coefficients

A fixed-point coefficient of wordlength W can be represented as a sum of SPT terms in the general form

$$h_m = \sum_{i=1}^{W} s_i 2^{-i} \qquad (7)$$

where $s_i \in \{-1, 0, 1\}$. Here we assume $-1 < h_m < 1$.

A minimum representation refers to the representation with the minimum required number of SPT terms. One minimum representation is the canonic signed digit (CSD) representation. Here, no two adjacent SPT terms can both be nonzero.

It is possible to transform the SPT representation in (7) to 0/1-variables as

$$h_m = \sum_{i=1}^{W} (x^+_{m,i} - x^-_{m,i}) 2^{-i} \qquad (8)$$

where $x^+_{m,i}, x^-_{m,i} \in \{0, 1\}$. This is advantageous as the formulation of the optimization objective function will be linear, otherwise it would have to be formulated as a nonlinear programming problem. Furthermore, it will make it possible to have linear constraints on the number of SPT terms per coefficient [7].

To obtain a CSD solution the following constraints can be added

$$x^+_{m,i} + x^-_{m,i} + x^+_{m,i+1} + x^-_{m,i+1} \le 1 \qquad (9)$$

for $m = 0, 1, ..., N/2$ and $i = 1, 2, ..., W - 1$. Adding these constraints yield a significant decrease of the solution time [7].

4.2. Normalized Peak Ripple Magnitude

The previous discussion assumed that the passband gain of the filter is equal to one. However, when the filter coefficients are expressed using a fixed-point format it is more convenient to allow an arbitrary passband gain. The stopband attenuation is then related to the passband gain. This introduces the opportunity to find "better" fixed-point coefficients, as it allows more flexibility [8].

The *normalized peak ripple magnitude* (NPRM) is commonly used for this type of problems. The passband gain is then computed as

$$g = \frac{\max\{|H_R(\omega T)|\} + \min\{|H_R(\omega T)|\}}{2} \qquad (10)$$

where ωT is in the passband of the filter.

If (10) is introduced in (6) the inequalities will be nonlinear, and, thus, it is not always possible to guarantee an optimal solution. Instead, we introduce a continuous scaling variable, s, and rewrite (6) as

$$-\delta(\omega T) \le \frac{H_R(\omega T)}{s} - D(\omega T) \le \delta(\omega T) \qquad (11)$$

The passband gain is always larger than zero. Hence (11) can be rewritten as

$$s[D(\omega T) - \delta(\omega T)] \le H_R(\omega T) \le s[D(\omega T) + \delta(\omega T)] \qquad (12)$$

to obtain a linear inequality with the passband gain as a variable.

The difference between the introduced form and the original NPRM is that in (12) the maximum absolute value of the passband ripple can be different between the positive and negative ripple.

It could be argued that (10) must hold, but if a filter design with a specified passband gain is considered, it is rarely the case that symmetric passband ripple is required. Hence, the proposed form not only provides linear inequalities, it could also be argued that it is a more natural way of expressing the passband gain. Indeed, (12) is just a scaled version of (6).

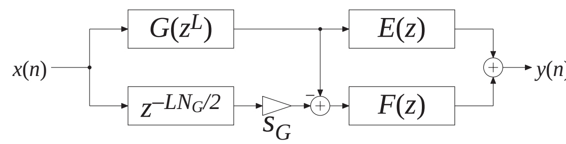

Figure 2. Proposed FRM filter structure for MILP design using NPRM.

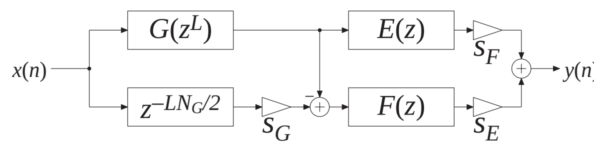

Figure 3. Proposed FRM filter structure for MILP design using NPRM with separate design of masking filters.

4.3. Problem Formulation

The zero-phase frequency response of an FRM filter can be written as

$$H_R(\omega T) = G_R(L\omega T)E_R(\omega T) + [1 - G_R(L\omega T)]F_R(\omega T) \quad (13)$$

It is clear that this formulation can not be used straightforward in the linear constraints. Instead, a two-stage approach must be used. In one of the stages the model filter, $G(z)$, is designed with $E(z)$ and $F(z)$ fixed. In the other stage, the masking filters, $E(z)$ and $F(z)$, is simultaneously designed with fixed $G(z)$. Note that (13) can be rewritten as

$$H_R(\omega T) = G_R(L\omega T)[E_R(\omega T) - F_R(\omega T)] + F_R(\omega T) \qquad (14)$$

which will be used later.

Considering the NPRM solution implicates two things for the FRM filters. First, as the complementary filter is obtained by subtracting the output of the model filter from a delay line, the passband gain of the model filter should be equal to that of the delay line. This can be obtained by either fixing the passband gain to one or by multiply one of the inputs to the subtraction. From a linear programming point of view, we propose that the output of the delay line is multiplied with the passband scaling gain of the model filter as shown in Fig. 2. This will lead to linear constraints, as opposed to if the scaling coefficient was introduced in cascade with $G(z)$. The introduced scaling coefficient is included in the cost function.

The second implication is that the passband gain of the two model filters must be equal. This can be treated either by having a continuous passband gain that is equal for both filters or by introducing fixed-point scaling variables as shown in Fig. 3. The latter case then requires that the three filters are designed separately, but at the same time it simplifies each design problem, leading to that higher order filters can be designed. Note that the scaling variables in Fig. 3 are introduced in the opposite branch to the corresponding filter to obtain linear constraints. We will only consider the first case here, but the second case can easily be derived similarly.

Denote the scaling variables s_G for the model filter and s_{EF} for the masking filters. Further, denote the filter coefficients e_m, f_m, and g_m, and the corresponding SPT terms, $x_{m,i}$, $y_{m,i}$, and $u_{m,i}$ for

$E(z)$, $F(z)$, and $G(z)$, respectively. The SPT terms for s_G are denoted v_i. Furthermore, let M_X and W_X denote the number of filter coefficients and coefficient wordlength for filter $X(z)$.

To minimize the number of SPT terms for $G(z)$ and s_G with fixed $E(z)$, $F(z)$, and s_{EF} the MILP problem in (15) should be solved.

$$
\begin{aligned}
&\text{minimize} \quad \sum_{m=1}^{M_G}\sum_{i=1}^{W_G} u_{m,i}^{+} + u_{m,i}^{-} + \sum_{i=1}^{W_s} v_i^{+} + v_i^{-} \\
&\text{subject to} \quad G_R(L\omega T)[E_R(\omega T) - F_R(\omega T)] + \\
&\qquad + s_G F_R(\omega T) \le s_G s_{EF}[D(\omega T) + \delta(\omega T)] \\
&\qquad G_R(L\omega T)[E_R(\omega T) - F_R(\omega T)] + \\
&\qquad + s_G F_R(\omega T) \ge s_G s_{EF}[D(\omega T) - \delta(\omega T)] \\
&\qquad g_m = \sum_{i=1}^{W_G} (u_{m,i}^{+} - u_{m,i}^{-})2^{-i}, \quad m = 1, 2, \ldots, M_G \\
&\qquad s_G = \sum_{i=1}^{W_s} (v_i^{+} + v_i^{-})2^{-i+k}
\end{aligned}
\tag{15}
$$

In a similar way, to minimize the number of SPT terms for $E(z)$ and $F(z)$ with fixed $G(z)$ and s_G, the MILP problem in (16) should be solved.

$$
\begin{aligned}
&\text{minimize} \quad \sum_{m=1}^{M_E}\sum_{i=1}^{W_E} x_{m,i}^{+} + x_{m,i}^{-} + \sum_{m=1}^{M_F}\sum_{i=1}^{W_F} y_{m,i}^{+} + y_{m,i}^{-} \\
&\text{subject to} \quad [s_G - G_R(L\omega T)]F_R(\omega T) + \\
&\qquad + G_R(L\omega T)E_R(\omega T) \le s_G s_{EF}[D(\omega T) + \delta(\omega T)] \\
&\qquad [s_G - G_R(L\omega T)]F_R(\omega T) + \\
&\qquad + G_R(L\omega T)E_R(\omega T) \ge s_G s_{EF}[D(\omega T) - \delta(\omega T)] \\
&\qquad e_m = \sum_{i=1}^{W_E} (x_{m,i}^{+} - x_{m,i}^{-})2^{-i}, \quad m = 1, 2, \ldots, M_E \\
&\qquad f_m = \sum_{i=1}^{W_F} (y_{m,i}^{+} - y_{m,i}^{-})2^{-i}, \quad m = 1, 2, \ldots, M_F
\end{aligned}
\tag{16}
$$

The solutions to these optimization problems yield a coefficient set that is minimal in the number of SPT terms given the other filter(s) when the NPRM is considered. This problem can be solved using standard MILP solvers using, e.g., using a branch-and-bound technique.

For the first iteration, the fixed filter(s) is designed using infinite precision coefficients. We have selected to fix $G(z)$ and solve (16) as our first step, to obtain a solution for $E(z)$ and $F(z)$. In the second step (15) is solved. However, this can be selected arbitrarily and may yield different solutions.

If a prescribed passband gain is required the values of s_G and s_{EF} can be fixed before the optimization starts. Note also that s_G and s_{EF} must be larger than 0 as this otherwise would yield an incorrect optimal solution with all variables equal to 0.

4.4. Reducing the Number of Variables

It is possible to reduce the number of variables before the actual optimization starts [7]. This is due to the fact that the filter coefficient values are small at the start and end of the impulse response.

Upper and lower bounds for the filter coefficients in the current optimization step can be determined for a fixed passband gain using linear programming. From these bounds the maximum and minimum value of the passband gain scaling coefficient, s, can be derived utilizing the complete dynamic range of the representation. Now, it is possible to derive the upper and lower bounds on the scaled values of the filter coefficients. When these bounds have been calculated it can be determined which SPT terms that can be fixed to a certain value, usually zero. This method is discussed in detail in [7].

To decrease the solution time it is also useful to have an upper bound for the objective function, which can be obtained from, e.g., a heuristic approach [9], or from previous design iterations.

4.5. Optimality

The filter(s) designed in each stage is optimal given the other filter(s). Taking all filters into concern, the result is only locally optimal as a joint optimization of all filters may give better results. However, this will not lead to a convex formulation, and, thus, there are no solutions methods, except for exhaustive search, that guarantee that an optimal solution is found.

For the first design iteration, the filter(s) to be fixed are designed using infinite precision coefficients. The results then depends on this initial solution. A number of methods has been proposed that designs infinite precision filters with a lower complexity than designing the subfilters separately [10], [11]. One approach to reduce the complexity further may be to first design the filter using one of these methods and then use the solution in the first iteration.

5. EXAMPLES

All problems in this section are solved using the general purpose MILP solver ILOG CPLEX [12]. As all previously proposed methods minimize the ripple given a number of SPT terms, comparisons can not be made. Further, for the high filter orders that are considered in the examples, the optimal MILP approach for direct realizations [7] is too time consuming.

The number of SPT terms given as result is the number required for the coefficients utilizing symmetry.

5.1. Example 1

In this example, a linear-phase FIR filter with passband edge at 0.47π rad and stopband edge at 0.5π rad is designed. The passband and the stopband ripples are 0.01.

A direct realization of this filter would require a filter order of 130, corresponding to 61 multipliers and 130 adders. Using an FRM filter with $L = 5$, we select $N_G = 32$, $N_E = 21$, and $N_F = 27$.

Using the proposed two-step design method the number of SPT terms for $E(z)$ and $F(z)$ after the first iteration are 13 and 17, respectively. The effective filter order of $E(z)$ is here 19 as the first and last coefficient is zero. Using these filters a solution with 28 SPT terms are found for $G(z)$ and s_G. Further iterations does not improve the results. In this case $s_G = 1$. The magnitude responses for the filters are shown in Fig. 4.

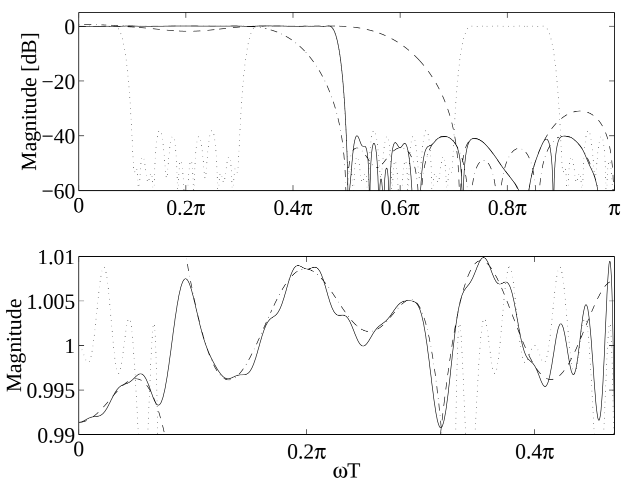

Figure 4. Magnitude responses for the filters in Example 1 (above) with passband (below). Total filter (solid), $G(z^L)$ (dotted), $E(z)$ (dashed), and $F(z)$ (dash-dotted).

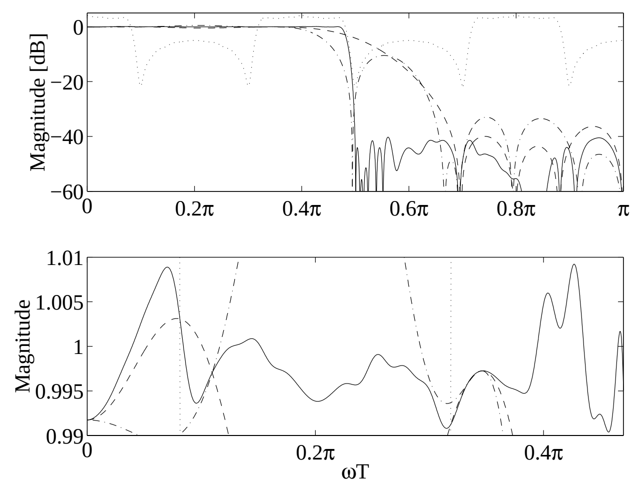

Figure 5. Magnitude responses for the filters in Example 2 (above) with passband (below). Total filter (solid), $G(z^L)$ (dotted), $E(z)$ (dashed), and $F(z)$ (dash-dotted).

5.2. Example 2

In this example, a filter with the same specification as in Example 1 is designed. However, here we first apply the optimization technique in [10] to obtain an initial filter with smaller ripple. This should lead to that the design margin is larger and that a solution with fewer SPT terms may be found. The same filter orders as in Example 1 is selected.

The number of SPT terms after the first iteration for $E(z)$ and $F(z)$ are 14 and 15, respectively. Hence, one SPT term is saved for $E(z)$ and $F(z)$. The effective filter order for $F(z)$ is now 23. For $G(z)$ a solution with 24 SPT terms is now obtained. Here, four SPT terms are saved. The passband gain is now 3/4, requiring two SPT terms, so the actual filter coefficients requires 22 SPT terms. Further iterations do not improve the solution. The magnitude responses for the filters are shown in Fig. 5.

This leads to that the number of SPT terms can be reduced using a different initial filter. It also indicates that optimization techniques for infinite precision filter coefficients may be yield a better initial filter as discussed in Section 4.5.

6. CONCLUSIONS

In this work a mixed integer linear programming (MILP) problem that minimizes the number of signed-power-of-two (SPT) terms given a filter specification for linear-phase frequency-response masking (FRM) filters was formulated.

The proposed method designs the filters in two steps. The model filter is designed given the masking filters and the masking filters is designed given the model filter. Hence, it is not guaranteed that the global minimum is obtained. However, each solution is minimal given the other filter(s), and iteration may improve the overall solution.

The filter design problems were formulated using normalized peak ripple magnitude (NPRM), which for FRM filters introduces some implications, which also were treated in this work.

Finally, two examples showed the viability and some properties of the proposed design approach.

7. REFERENCES

[1] T. Saramäki and J. Yli-Kaakinen, "Design of digital filters and filter banks by optimization: applications," *Proc. X European Signal Processing Conf.*, Tampere, Finland, Sept. 4–8, 2000.

[2] L. Wanhammar and H. Johansson, *Digital Filters*, Linköping University, 2002.

[3] Y. C. Lim, "Frequency-response masking approach for the synthesis of sharp linear phase digital filters," *IEEE Trans. Circuits Syst.*, vol. 33, pp. 357–364, Apr. 1986.

[4] Y. Lian, "Design of discrete valued coefficient FIR filters using frequency response masking," in *Proc. Int. Conf. Electronics, Circuits, Syst.*, Paphos, Cyprus, Sept. 5–8. 1999, pp. 253–255.

[5] W. R. Lee et al., "Frequency-response masking based FIR filter design with power-of-two coefficients and optimum PWR," in *Proc. IEEE Int. Symp. Circuits Syst.*, Phoenix, AZ, 2002, pp. 1175–1178.

[6] Y. J. Yu and Y. C. Lim, "Genetic algorithm approach for the optimization of multiplierless sub-filters generated by the frequency-response masking technique," in *Proc. Int. Conf. Electronics, Circuits, Syst.*, Dubrovnik, Croatia, Sept. 15–18, 2002, vol. 3, pp. 1163–1166.

[7] O. Gustafsson, H. Johansson, and L. Wanhammar, "An MILP approach for the design of linear-phase FIR filters with minimum number of signed-power-of-two terms," in *Proc. European Conf. Circuit Theory Design*, Espoo, Finland, Aug. 28–31, 2001.

[8] Y. C. Lim, "Design of discrete-coefficient-value linear phase FIR filters with optimum normalized peak ripple magnitude," *IEEE Trans. Circuits Syst.*, vol. 37, pp. 1480–1486, Dec. 1990.

[9] O. Gustafsson, *Contributions to Low-Complexity Digital Filters*, Linköping Studies in Science and Technology, Diss., No. 837, Linköping University, Sweden, 2003.

[10] T. Saramäki and H. Johansson "Optimization of FIR filters using the frequency-response masking approach," in *Proc. IEEE Int. Symp. Circuits Syst.*, Sydney, Australia, May 6-9, 2001, vol. II, pp. 177–180.

[11] W. S. Lu and T. Hinamoto, "Optimal design of frequency-response-masking filters using semidefinite programming," *IEEE Trans. Circuits Syst.*, vol. 50, pp. 557–568, Apr. 2003.

[12] ILOG CPLEX, http://www.ilog.com/products/cplex/, 2003.