High Accuracy Design and Realization of Fixed and Variable Allpass-based Fractional Delay Digital Filters

Georgi K. Stoyanov, Kamelia S. Nikolova, Valentina I. Markova

Abstract – In this paper new methods for low-sensitivity design of allpass-based fixed and variable fractional delay filters are proposed. They are based on a sensitivity minimization of each individual first- and second-order allpass section in the cascade realization depending on their transfer function pole positions. Thus a higher accuracy of implementation and tuning, shorter wordlength and efficient multiplierless realizations are achieved.

Keywords – Digital Filters, Fractional delay filters, Allpass filters, Low sensitivity, High Accuracy.

I. INTRODUCTION

Conventional linear digital circuits are providing usually a delay response that is equal to an integer number of sampling intervals (as in linear-phase FIR (finite-impulse-response) realizations) or is changing uncontrollably with the frequency (for all IIR (infinite-impulse-response) digital filters). It appeared, however, that we might often need a circuit with a delay response that is a fraction of the sampling interval and is fixed or variable (or only adjustable). Design and implementation of such circuits with given and properly controlled fractional delay (FD) is the hottest digital filters topic in the last ten years. These circuits are invaluable in many telecommunications applications, like time adjustment and precise jitter elimination in digital receivers, echo cancellation, phase-array antenna systems, transmultiplexers, sample-rate converter and software radio. They are needed in speech synthesis and processing, image interpolation, sigma-delta modulators, time-delay estimation, in some biomedical applications and for modeling of musical instruments. Most of these applications are overviewed in Refs. [1], [2].

The design of fixed FIR FD filters (FDF) is well developed and quite mature field, because it is relatively easy to formulate the design problem and to obtain an optimal solution. Many methods, so far, have been advanced and most of them are well summarized in [1], [2]. They include a least squared (LS) integral error design, often combined with properly selected window functions or other methods for smoothing the filter transition band; weighted LS integral error approximation of the frequency response [1]; maximally-flat FD design based on Lagrange interpolation (very popular and widely used, but with several drawbacks [3]); minimax design, achieving lower than LS and Lagrange filters maximal error [2]; splines-based FDF design [1], generalized in [4]. Most of these methods are used to design also variable FD (VFD) FIR filters. The most popular structure for such filters is the famous Farrow structure [5]. There are many other design methods like a constrained minimax optimization method [6], a singular value decomposition method [3], a Taylor series expansion method [7], and the weighted LS design [8]. Most of them are using the Farrow structure or its modifications [9]. Common disadvantages of all the FIR FDFs are their higher complexity (higher order transfer function (TF) and too many multipliers and delays), very high overall delay and not constant for all frequencies magnitude response, varying additionally when the delay is tuned.

Recently, several methods for design and implementation of general IIR variable FDFs have been proposed [10], [11]. The method in [10] is based on a two-stage procedure, where in the first stage a set of fixed delay general IIR filters are designed by minimizing a quadratic objective function defined by integrated error criterion; in the second stage the TF coefficients of the fixed delay filters are represented as polynomials and are fitted for any given FD. The method in [11] is based on a new model reduction technique and is applicable to IIR TFs that are decomposable to sub-filters with a common denominator, realized then as Farrow structures. Both methods are achieving an impressive FD variability, but at a price of too higher TF order (30 or 55 in [10]) and calculation of too many multiplier coefficients (for example 426 in [10]), to be practical. There are, however, IIR FDFs (fixed and variable), avoiding all the disadvantages of the FIR and of the general IIR FDFs, and they are based on allpass structures. The main advantage of the allpass-based FDF is that their magnitude is unity for all frequencies and it remains unity when the FD is tuned. The TF order of these filters is low and so are the circuit complexity and the total delay time compared to those of the FIR realizations. Many methods for design of allpass based FDF have been described in [1], [2] and many more new methods (mainly for variable FDFs) have been proposed after that. One group in [1], [2] consists of several weighted LS (WLS) methods. Recently [12] a new iterative WLS method was developed, but it was shown [13] that very often it is not converging. A new noniterative approach solving the minimization problem by solving a matrix equation and thus avoiding the convergence problems was advanced in [13]. Both methods are rigorously proven and are producing very impressive results (very low frequency response error), but as with the general IIR methods, the TF order is very high (35 for example), each of the multiplier coefficients is represented by polynomial of 5th or 6th order (making thus the total number of the coefficients more than 200). Then 100 sets of coefficients are calculated to cover the frequency range from 0 to 0.9π, and another 30 sets are calculated to cover the range of FD from -0.5 to 0.5. And if the required FD if not coinciding with some of these 30 sets,
new coefficients are calculated throughout the polynomial interpolation. All this is making that approach to realize allpass-based VFDFs quite unpractical and difficult to tune them in real time. Another group of design methods encompasses all the minimal approaches to allpass FDFs design in terms of minimal phase error, phase-delay or group-delay error [1]. An improved optimization method was proposed in [14] to overcome the problems with the convergence when designing VFDFs. It is based on gradual increase of the filter order and optimization in minimal sense to obtain optimal values for the adjustable parameters. This method is addressing the famous “gathering structure” [15], widely used for realization of allpass-based VFDFs. The third and most popular group of methods is the maximally-flat design of allpass FDFs based on Thiran approximation [16], giving a closed-form solution for the TF coefficients. The Thiran-based design of VDF is somehow connected to the gathering structure, which permits very easy real-time tuning by recalculating and reprogramming a single coefficient value. This structure was criticized recently for its long critical path and big difference between the coefficient values (requiring longer wordlength) and an improved structure was proposed in [17]. Another way to use Thiran approximation but to avoid usage of gathering structure to realize VDF (and thus to avoid the division operation in the recalculation of the coefficients) was proposed in [18] and it is called “root displacement interpolation (RDI) method” (See Sect. VI.A).

The main aim of the present work is to investigate and compare the existing and to develop new methods of realization and tuning of allpass-based FDFs and to increase the accuracy throughout minimization of their sensitivities. It will permit more efficient multiplierless realizations, shorter wordlength and lower power consumption.

II. HIGH-ACCURACY DESIGN PRINCIPLES

It is clear from the above considerations that allpass based FDFs (with fixed and variable FD) are most appropriate for almost all practical applications, providing lower order TF, low complexity and low total delay-time realizations, permitting an easy real-time FD tuning. The strategy to achieve our aim is based on our approach, described in [19] and using (when possible) a cascade realization of the allpass TF. In order to further decrease the overall sensitivity, we propose, after decomposing the allpass TF to first- and second-order terms, to minimize the sensitivities of the individual first- and second-order allpass sections, realizing each real pole or couple of complex-conjugate poles. This minimization may consist of careful selection of proper sections (there are too many allpass sections already known) according to the position of the poles in the z-plane or of development of new allpass sections when there is no low sensitivity realizations readily available for given pole positions.

We select to use the Thiran approximation procedure [16] for designing allpass based FD digital filters with maximally flat group delay response. This procedure gives an easy way to express the TF coefficients \( a_k \) as a function of the desired fractional delay parameter value \( D \).

\[
a_k = (-1)^k \frac{(N)}{D-N+n} \prod_{k=0}^{N} \frac{D-N+k+n}{D-N+k+n}, \quad \text{for } k = 0, 1, 2 \ldots N, \quad (1)
\]

for every allpass TF of \( N \)-th order

\[
H_{AP}(z) = \frac{a_N + a_{N-1}z^{-1} + \ldots + a_1z^{-N-1} + a_0z^{-N}}{a_0 + a_2z^{-1} + a_2z^{-2} + \ldots + a_Nz^{-N}} = \frac{B(z)}{A(z)}. \quad (2)
\]

In the literature very often this allpass TF is realized as a direct form \((2N + 1 \text{ multipliers and } N \text{ delays are needed for the realization})\), which is by far non-canonic with respect to the multipliers number (a canonic allpass structure of \( N \)-th order should contain only \( N \) multipliers) and is very sensitive to the changes of the coefficient values. It is well known that a cascade realization of the allpass TF will decrease considerably the overall sensitivity and will open the way for further sensitivity reduction as already mentioned.

III. ALLPASS TRANSFER FUNCTIONS POLES LOCI INVESTIGATIONS

The sensitivities of the realizations are strongly depending on the position of their poles in the z-plane, so it is important to know how the poles of the allpass-based FD filters are situated there.

A. Real Poles Behavior

The possible real poles positions can be split into 3 cases:

1. Odd order FD TF and \( N-1 < D < N \) - the real pole is negative. When the FD parameter values are increasing from \( N-1 \) to \( N \), the possible poles positions are moving from \( z = -1 \) to the area near \( z = 0 \) (as case 1 in Fig. 1).

2. Odd order FD TF and \( D > N \) - the real pole is positive and increasing \( D \) to infinity moves the pole from the area near \( z = 0 \) to the area near \( z = 1 \) (as case 2 in Fig. 1).

3. Even order FD TF and \( N-1 < D < N \) - there exist one negative real pole and one positive real pole as shown in the Fig. 1 for sixth order FD TF. When the FD is increasing from \( N-1 \) to \( N \), these two possible poles are moving as in the above mentioned cases 1 and 2.

B. Complex-conjugated Poles Behavior

The complex-conjugated poles behavior falls into two categories regarding the range of the FD parameter values.

1. \( N-1 < D < N \) - the complex-conjugated poles pairs are situated around the area \( z = 0 \) and can be either with positive or negative real part depending of a given FD parameter value as can be seen from Fig. 1.

2. \( D > N \) - the behavior of the poles is more dynamical. The complex-conjugated poles are positioned mainly in the right half of the unit circle and only the higher order TFs have poles in the left half, as illustrated in Fig. 1. The dashed line with number 3 shows the poles movement for increasing the FD parameter values to infinity.
Fig. 1. Possible poles position of real poles (for odd-order TF) and of all the poles of sixth order allpass FD TF

IV. ALLPASS SECTIONS SENSITIVITIES STUDY

A. First Order Allpass FD Sections

It follows from Fig. 1 that if a cascade realization of the FD allpass filters would be used, as the possible real pole positions are scattered all around the real axes, first-order allpass sections with low sensitivities for all these positions will be needed. About 20 such sections have been investigated and compared in [20] and it was shown that several low-sensitivity sections for every real pole-position could be found. We select to use four of them, namely the ST1 section, providing low-sensitivity for poles near $z=1$, MH1 and SC, having low sensitivity for poles near $z=0$ and SV section for poles near $z=-1$. Their TFs are:

$$ H_{MH1}(z) = \frac{-b + z^{-1}}{1 - bz^{-1}}, \quad (3) $$

$$ H_{SC}(z) = \frac{-b - z^{-1}}{1 + bz^{-1}}, \quad (4) $$

$$ H_{ST1}(z) = \frac{-(1-a) + z^{-1}}{1 - (1-a)z^{-1}}, \quad (5) $$

$$ H_{SV}(z) = \frac{1 - c + z^{-1}}{1 + (1-c)z^{-1}}. \quad (6) $$

The closed form solutions for their TF coefficients and the FD parameter $D$ (1) are:

$$ b_{MH1} = \frac{D-1}{D+1}; \quad b_{SC} = -\frac{D-1}{D+1}; \quad (7) $$

$$ a_{ST1} = \frac{2}{D+1}; \quad c_{SV} = \frac{2D}{D+1}. \quad (8) $$

In Fig. 2 the worst-case phase-response-sensitivities of these four sections are given for realizations with different TF pole positions. It is clearly seen that there exists a proper choice of sections for every possible pole position and the difference between the maximal values of the sensitivities may reach 10 times.

B. Second Order Allpass FD Sections

There are a great number of second order allpass sections in the literature and we need some preliminary selection among them before starting deeper study. The complex-conjugated poles are positioned mainly in the right half of the unit circle and only rarely (for higher TFs order) in the left half, as illustrated in Fig. 1. Our extensive investigations show that the study, the classification and the selection of second order allpass sections will be eased if those complex-conjugated poles are grouped into 11 zones as shown in Fig. 3 for the upper half of the unit circle. The poles positions of tenth order allpass based FD filter, for example, for values of $D$ in the range $N<D<50$ will scatter as shown in Fig. 3, but for $N<D<N+1$ (the most typical case) they all will concentrate only in zones 1, 2, 5, 6. This is valid also for TFs of any order. Thus, we will need most often second-order allpass sections with minimized sensitivities for complex-conjugated poles pairs positioned in these zones in order to obtain low-sensitivity FD realization and better FD time accuracy. These zones are not typical for conventional selective filters, whose poles are situated usually near $z=1$, so we selected initially the most popular sections, having canonic structures and known with low sensitivities [21], [22]. They are the Gray-Markel section...
(GM2), the Mitra and Hirano sections (MH2A and MH2B), the Kwan sections (KW2A and KW2B) and the low sensitivity section ST2A, with the following TFs (without ST2A):

\[ H_{\text{MH2A}}(z) = \frac{b_1 b_2 - b_1 z^{-1} + z^{-2}}{1 - b_1 z^{-1} + b_2 z^{-2}} \]  

(9)

\[ H_{\text{MH2B}}(z) = \frac{b_2 - b_1 z^{-1} + z^{-2}}{1 - b_1 z^{-1} + b_2 z^{-2}} \]  

(10)

\[ H_{\text{GM2}}(z) = \frac{-a_1 - a_2 (1-a_1) z^{-1} + z^{-2}}{1 - a_2 (1-a_1) z^{-1} - a_1 z^{-2}} \]  

(11)

\[ H_{\text{KW2A}}(z) = \frac{1 + a_1 - a_2 - (a_1 + a_2) z^{-1} + z^{-2}}{1 - (a_1 + a_2) z^{-1} + (1 + a_1 - a_2) z^{-2}} \]  

(12)

\[ H_{\text{KW2B}}(z) = \frac{d_1 + d_2 - \left(-d_1 - d_2 \right) z^{-1} + z^{-2}}{1 - (d_1 - d_2) z^{-1} + (d_1 + d_2 - 1) z^{-2}} \]  

(13)

It appeared, however, that all these sections, developed for selective filters applications, are not having enough low sensitivities for poles in zones 1, 2, 5, 6, as shown in Fig. 5, where especially wrong choice is ST2A. We have developed in [23] a new section, shown in Fig. 4 (we shall call it IS-section), with minimized sensitivity for the TF poles situated exactly in zone 2. Its transfer function is

\[ H_{\text{IS}}(z) = \frac{b + (-a - 2b + ab) z^{-1} + z^{-2}}{1 + (-a - 2b + ab) z^{-1} + b z^{-2}} \]  

(14)

![Fig. 4. IS allpass section, suitable for FD filter realizations with TF poles in zone 2](image)

It is canonic with respect to the number of the multipliers and the delays, its round-off noises are constant and very low and it is structurally lossless and structurally bounded real.

The phase sensitivities of the new allpass section together with those of the other second-order allpass sections were investigated for complex-conjugated poles pairs in the zones 1, 2, 5 and 6. The results for the worst-case phase sensitivities are given in Fig. 5. It is obvious that the worst case phase sensitivity of the IS section is the lowest for small values of the FD parameter $D$ which correspond to TF poles situated in zone 2. The other allpass sections suitable for realizations of small values of FD are GM2 and MH2B (zone 6) and GM2 and MH2A (zone 1 and zone 5), KW2A and KW2B (not shown in Fig. 5) and ST2A generally cannot be recommended and have to be investigated in every specific case. The TF coefficients as function of $D$ (1) are given in Tables I–III.

**TABLE I. IS AND GM2 FD TRANSFER FUNCTION COEFFICIENTS**

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<thead>
<tr>
<th></th>
<th>IS</th>
<th>GM2</th>
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<tbody>
<tr>
<td>$a$</td>
<td>$(D^{-2})$</td>
<td>$(D^{-1}D^{-2})$</td>
</tr>
<tr>
<td>$b$</td>
<td>$(D+1)(D+2)$</td>
<td>$(D-1)(D+2)$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$(D-1)(D-2)$</td>
<td>$(D-2)(D+2)$</td>
</tr>
<tr>
<td>$a_2$</td>
<td>$(D+1)(D+2)$</td>
<td>$(D+1)(D+2)$</td>
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</table>

**TABLE II. MH2A AND MH2B FD TRANSFER FUNCTION COEFFICIENTS**

<table>
<thead>
<tr>
<th></th>
<th>MH2A</th>
<th>MH2B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_1$</td>
<td>$(D^{-2})/(D+1)$</td>
<td>$(D^{-1})/(D+2)$</td>
</tr>
<tr>
<td>$b_2$</td>
<td>$(D^{-2})/(D+1)$</td>
<td>$(D^{-1})(D^{-2})/(D+1)(D+2)$</td>
</tr>
<tr>
<td>$b_1$</td>
<td>$(D^{-1})/(D+1)$</td>
<td>$(D^{-2})/(D+1)$</td>
</tr>
<tr>
<td>$b_2$</td>
<td>$(D^{-1})(D^{-2})/(D+1)(D+2)$</td>
<td>$(D^{-1})(D^{-2})/(D+1)(D+2)$</td>
</tr>
</tbody>
</table>

**TABLE III. KW2A AND KW2B FD TRANSFER FUNCTION COEFFICIENTS**

<table>
<thead>
<tr>
<th></th>
<th>KW2A</th>
<th>KW2B</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
<td>$(D^{-2}D^{-1}D^{-4})/(D+1)(D+2)$</td>
<td>$(D^{-1}D^{-3}D^{-4})/(D+1)(D+2)$</td>
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<tr>
<td>$a_2$</td>
<td>$(D^{-2})/(D+1)$</td>
<td>$(D^{-1})(D^{-2})/(D+1)(D+2)$</td>
</tr>
<tr>
<td>$d_1$</td>
<td>$(D^{-2})/(D+1)$</td>
<td>$(D^{-1})(D^{-2})/(D+1)(D+2)$</td>
</tr>
<tr>
<td>$d_2$</td>
<td>$(D^{-1})(D^{-2})/(D+1)(D+2)$</td>
<td>$(D^{-1})(D^{-2})/(D+1)(D+2)$</td>
</tr>
</tbody>
</table>
V. HIGH ACCURACY DESIGN OF FIXED FD FILTERS

Having in mind the principles of high-accuracy design from Sect. II and taking into account the results obtained hereabove, we propose the following design procedure: 1. Apply the Thiran approximation to obtain an allpass TF with order \( N \) ensuring a phase-delay error within given limits over the required frequency range. Broadening excessively this range will increase considerably the order \( N \). 2. Decompose the TF to first and second-order terms and check in which zones the poles of these terms are situated. 3. Select first and second-order allpass sections providing lowest sensitivities for each real or couple of complex-conjugated poles. 4. For poles in some zones, as seen in Figs. 2 and 5, several allpass sections are equally good possible candidates. In such case compose some zones, as seen in Figs. 2 and 5, several allpass sections and investigate the overall sensitivity of each set to select the one with the lowest sensitivity. This procedure was applied to obtain an FD allpass structure realizing an 11th order allpass-based FD TF with \( D = 11.2 \). The 11th order TF has five pairs of complex-conjugated poles (two pairs in zone 1 and three in zone 2) and one real pole, as shown in Fig. 6. The most recommendable (from what follows from Figs. 2 and 5) set of allpass sections is suggested in the same figure, but the other possible four sets have also been considered. The worst-case phase sensitivities of the realizations, corresponding to all the five sets, are shown in Fig. 7.

It is seen from Fig. 7 that the method is working properly and two of the sets are by far worst than the other three. It is amazing, that for this specific example, there are three sets of allpass sections that are having very similar overall worst-case sensitivity and the final choice has to be made after considering other details, like total number of adders, range of values of multiplier coefficients and deterioration of the delay response after the coefficients quantization.

VI. HIGH ACCURACY VARIABLE FD FILTERS

A. Design Procedures

The calculation of the coefficients obtained by Thiran approximation (1) include too many division operations that are making difficult tuning of such circuit in real time. In [15] the coefficients (1) have been presented as:

\[
\hat{a}_k = (-1)^k \left( \sum_{n=0}^{N} \frac{(d+n)}{\prod_{m=1}^{k-1}(d+n+m)} \right) \prod_{n=k+1}^{N}(d+N+n) =
\]

\[
(-1)^k \left( \frac{N}{k} \sum_{n=0}^{N-k} \frac{(d+n)}{\prod_{m=1}^{k-1}(d+n+m)} \right) \prod_{n=k+1}^{N}(d+N+n) =
\]

\[
= \sum_{k=1}^{N} \hat{a}_k d^k, \text{ for } k=1,\ldots,N,
\]

where \( d \) is the fractional part of the phase-delay.

Then, the allpass TF (2) was given in the form

\[
H_{AP}(z) = \frac{g(d)\hat{a}_N + \cdots + \hat{a}_1 z^{-(N-1)} + z^{-N}}{1 + g(d)\hat{a}_1 z^{-1} + \cdots + \hat{a}_N z^{-N}},
\]

and the coefficient \( g(d) \) was approximated using the truncated Maclaurin series as

\[
g(d) = \frac{1}{\prod_{n=1}^{N}(d+N+n)} = \left[ 1 + \sum_{k=1}^{N} (-1)^k \left( \frac{d}{N+n} \right)^k \right] \equiv \sum_{i=0}^{N} g_i d^i,
\]

where \( I \) is the order of the approximating polynomial. The structure obtained through this method is called “gathering structure”. Even though very famous, this structure has many drawbacks: (a) it contains a great number of multipliers and adders leading to long critical paths; (b) as any direct structure is suggested in the same figure, but the other possible four sets have also been considered. The worst-case phase sensitivities of the realizations, corresponding to all the five sets, are shown in Fig. 7.

Fig. 6. Pole-position plot of 11th order allpass FD filter realizing \( D = 11.2 \)

Fig. 7. Worst-case phase-sensitivities of different sets of sections realizing an 11th order allpass-based FD TF with \( D = 11.2 \).
and shorter critical path, compared to gathering structure, and values of the coefficients $c_{mn}$ (18) of the same order.

We found in [24], [25] that it is possible to obtain even more efficient variable realizations by expressing each transfer function coefficients $a_k$ (2) as Taylor series expansion with respect to $d$ and then to truncating after the linear, square or cubic term ($T=1, 2, 3$) depending on the desired accuracy. To achieve the tuning in real time we propose the following design procedure:

1. Select the allpass TF order corresponding to given requirements (desired fractional delay value $D$ and/or the bandwidth with maximally flat phase delay response).
2. Obtain an allpass FD filter using Thiran approximation.
3. Taylor series expansion of each TF coefficient and truncation after the linear (when only adjustment of the phase delay is required), square or cubic term (if tuning over larger range of values of the phase delay is required).
4. Realize all the multiplier coefficients as composite multipliers (see Figs. 8, 9).

The proposed design procedure is simple to use and the obtained structures have no critical path. The method can be applied for an arbitrary TF order but in the cases of first and second order TFs it allows to implement structures different form direct form and to minimize the sensitivity of the realizations. For the low-sensitivity structure IS (Fig. 4), for example, the coefficients are expressed by $d$ as

$$a = \frac{d}{d+2}; \quad b = \frac{d(d+1)}{(d+3)(d+4)}.$$

After expanding (19) to Taylor series and truncate after square or cubic term, we get correspondingly:

$$a = \frac{1}{2} d - \frac{1}{4} d^2; \quad b = \frac{1}{12} d + \frac{5}{144} d^2;$$

$$a = \frac{1}{2} d - \frac{1}{4} d^2 + \frac{1}{8} d^3; \quad b = \frac{1}{12} d + \frac{5}{144} d^2 - \frac{47}{1728} d^3.$$

All these coefficients have homogenous structure, they do not include division operation and can be realized as composite multipliers containing fixed and variable multipliers. The composite multiplier realizations for second and third order Taylor approximation of $a$ are shown in Fig. 8 and Fig. 9.

![Fig. 8. Composite variable multiplier realization of $a$ (20) after a second order Taylor approximation](image)

![Fig. 9. Composite variable multiplier realization of $a$ (21) after a third order Taylor approximation](image)

The RDI-method [18], mentioned in the Introduction, is using two $N^{th}$ order allpass FD TFs approximating different FD values $D_1$ and $D_2$ to obtain a new allpass FD filter with phase delay time $D_i$ such that $D_1 < D_i < D_2$. The denominator of (2) (the denominators of the two initial allpass transfer functions) is represented as [18]:

$$A_i(z) = \left[1 - r_i z^{-1}\right] \prod_{k=1}^{(N-1)/2} \left[1 - c_{i,k}^2 z^{-2}\right], \quad N \text{ odd}$$

$$A_i(z) = \prod_{k=1}^{N-1} \left[1 - c_{i,k}^2 z^{-2}\right], \quad N \text{ even}$$

where $\{c_{i,k}, c_{i,k}^*\}$ is $k^{th}$ complex-conjugated pole pair and $r_i$ is the real pole of the filter with TF $H_i(z)$ (2). The complex-conjugated poles (for real pole is the same procedure) are sorted with respect to their angles and are paired according to their angular proximity. The interpolated complex poles can be calculated from the paired poles as

$$c_{\text{int},k} = \left[1 - \rho\right] c_{i,k} + \rho c_{2,k}^*,$$

where $\rho$ is constant between 0 and 1. This can be realized using only adders and multipliers, as shown in [18], and phase-delay time $D_i$ can be tuned within the range $D_1 < D_i < D_2$ by trimming only the constant $\rho$. This method is not connected to any particular realization of the initial allpass filters of order $N$, so the sensitivity cannot be an object of consideration in this case. Two disadvantages are readily seen, however: quite complicated circuitry (two allpass filters plus four additional multipliers) and narrow range of tuning of $D$ with growing error of tuning in the middle of this range.

### B. Accuracy Investigations

To compare the accuracy of the first three methods, considered in Sect. VI.A, we have designed and investigated realizations and tuning in the range $1.5 < D < 2.5$ (i.e. $d = \pm 0.5$ ) of second order allpass FD filters. For the polynomial approximation of the TF coefficients truncation after the third order term was used, i.e. $I = 3$ (17), $P = 3$ (18) and $T = 3$. The circuit-diagrams for the first two methods are given in Figs. 10 and 11. For our method, the IS-section (Fig. 4) with composite multipliers was used. The values of the coefficients of the three realizations are given in Table IV, Table V, and Eq. (21), correspondingly.

<table>
<thead>
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<td>TF COEFFICIENTS OF GATHERING STRUCTURE</td>
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<td>$a_k$</td>
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<td>$\hat{e}_{11} = -8$</td>
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<tr>
<td>$\hat{e}_{21} = -2$</td>
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<td>$\hat{e}_{31} = 1$</td>
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TABLE V

<table>
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<th>$c_{11}$</th>
<th>$c_{21}$</th>
<th>$c_{31}$</th>
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<tr>
<td>0.083333</td>
<td>0.034722</td>
<td>-0.027199</td>
</tr>
</tbody>
</table>

In Fig. 12 the worst-case phase sensitivities of the three realizations for several values of the fractional part $d$ of the phase-delay time are given. It is seen that our approach and the Cho-Parhi method are decreasing considerably the sensitivity, compared to that of the gathering structure, for $d = \pm 0.5$ (our structure is behaving better than that of Cho-Parhi for positive values of $d$ and it is opposite for the negative values). For small values of $d$ our structure is the best, but generally the IS and the Cho-Parhi structures are having similar sensitivities. The possible explanation for this is that the Cho-Parhi approach, when reducing the range of values of the multiplier coefficients, compared to those of the gathering structure, is decreasing the largest values. It is well known, that when the values of the multiplier coefficients are decreased, the sensitivities to these coefficients are decreased too.

In Table VI the complexities of the three variable realizations are compared. The Cho-Parhi structure is with the least number of multipliers, but our structure has only two delays, it is not having a critical path and it will be shown in the Experiments that it is behaving better in a limited wordlength environment.

<table>
<thead>
<tr>
<th>Variable IS structure</th>
<th>Gathering structure</th>
<th>Cho-Parhi structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 3$</td>
<td>$I = 3$</td>
<td>$P = 3$</td>
</tr>
<tr>
<td>Multiplier</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>Adder</td>
<td>14</td>
<td>9</td>
</tr>
<tr>
<td>Delay element</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Fig. 12. Worst-case phase-sensitivities of 2nd order allpass based FD filter ($I = 3$, $P = 3$, $T = 3$)

The RDI-method is not considered here, as it is not connected to some specific realization. Its accuracy is investigated in the Experiments (Sect. VIII).

VII. MULTIPLIERLESS REALIZATIONS OF ALLPASS BASED FD FILTERS

The reduction of the overall sensitivity permits shortening of the coefficients wordlength followed by more efficient multiplierless implementation. We have applied this approach in [26] and after deriving closed form expressions for the coefficients of the allpass sections given in Sect. IV, we have obtained multiplierless realizations with no more than three adders per coefficient. We can further improve the design by applying a genetic algorithm [27] to optimize the values of the coefficients within the set possible values limited by the quantization.

Let the phase delay response of (2) be:

$$\omega \rightarrow \arg \left( H_d(e^{j\omega}) \right)$$

and let the desired allpass filter TF be:

$$H_d(e^{j\omega}) = e^{-j\omega D} \quad \text{for} \quad \omega \in [0, \pi]$$

with an ideal phase responses

$$\arg H_d(e^{j\omega}) = -\omega D.$$  

It is known that for $D > N$ the phase delay response of stable allpass filter decreases monotonously from $D$ to $N$ as $\omega$ increase from 0 to $\pi$. Therefore, the aim is to approximate the desired phase delay response on the frequency band given
by \( \Omega_1 \in [0, \omega_1] \), where \( \omega_1 < \pi \). The phase delay error can be determined by:

\[
\max_{\omega \in \Omega_1} |f_{\omega}(\omega) - D|.
\]

(27)

The coefficient values for multiplierless realization are expressed as

\[
\sum_{r=1}^{R} a_r 2^{-P_r},
\]

(28)

where each \( a_r \) is either 1 or -1 and the \( P_r \)'s are nonnegative integers in the increasing order. In this case, the aim is to find all the coefficient values so that, first, \( R \), the number of powers-of-two terms, is made as small as possible, and, secondly, \( P_r \), the maximum number of shifts, is made as small as possible.

An estimate for the implementation cost of the filter can be calculated as a sum of the number of the adders and substracters used to implement all the filter coefficients, that is, the cost is given by

\[
\sum_{i=1}^{N} \sigma_i,
\]

(29)

where the \( \sigma_i \)'s are the number of adders and substracters required to implement the filter coefficients \( c_i \).

If \( \Phi(c) \) is the vector of the adjustable filter parameters subject of optimization, and the allpass TF (2) is presented as \( H_{\text{ip}}(\Phi, z) \), we can define the following optimization problem.

**Optimization Problem:** Given \( D \), find the adjustable parameter vector \( \Phi \) such that, first, the phase delay error Eq. (27) is minimized after quantizing the coefficient values according to the above-mentioned form for their representations and, then, the implementation cost (29) is minimized too.

The solution can be found in the following two-step procedure [28]. First, the smallest and the largest values of the FD filter coefficients for fractional delay parameter values varying in a given limits are determined such that the requirement are satisfied. This enables one to find the parameter space of the infinite-precision coefficients including the feasible space where the phase delay error will be minimized. The second step involves finding the filter parameters in this space such that the resulting FD allpass filter meets the given criteria with the simplest coefficient representation forms.

The number of discrete coefficient value combinations can be huge. For this reason it is beneficial to use a genetic algorithm for searching those discrete coefficient values with which the requirements are satisfied. First, the indexes of the power-of-two numbers between the smallest and largest values of the coefficients are represented using a binary code. Furthermore, a lookup table containing the power-of-two values of the corresponding indexes is generated. The next step is to construct the chromosomes by concatenating all these binary strings. At the end, the fitness of the population is evaluated by decoding the chromosomes to their corresponding power-of-two coefficients values using the above-mentioned lookup table.

The fitness function to be minimized is given by

\[
f = \min_{\omega \in \Omega_1} \max_{\omega \in \Omega_1} |f_{\omega}(\omega) - D|.
\]

(30)

Applying this algorithm on the second-order sections from Sect. IV, realizing FD time \( D=2.82 \), produced the coefficient values for multiplierless realization, shown in Table VII. All the coefficients are realized with maximum number of only two power-of-two terms \( (R=2) \), which means only two adders or substractors per coefficient, while the phase-delay error is remaining within the given limits. One of the sections (GM2), however, is losing its FD maximally-flat behavior after the optimization.

<table>
<thead>
<tr>
<th>Section</th>
<th>( c_1 (a_1, a_1, b_1) )</th>
<th>( c_2 (b, a_2, b_2) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>IS</td>
<td>( 2^{-2} + 2^{-5} )</td>
<td>( 2^{-4} + 2^{-6} )</td>
</tr>
<tr>
<td>GM2</td>
<td>( 2^{-4} )</td>
<td>( 2^{-1} - 2^{-3} )</td>
</tr>
<tr>
<td>MH2A</td>
<td>( 2^{-1} - 2^{-3} )</td>
<td>( 2^{-2} - 2^{-4} )</td>
</tr>
<tr>
<td>MH2B</td>
<td>( 2^{-1} - 2^{-3} )</td>
<td>( 2^{-4} - 2^{-7} )</td>
</tr>
</tbody>
</table>

Similar results have been obtained for the first-order allpass sections from Sect. IV.A.

**VIII. Experiments**

In order to verify the proposed low-sensitivity design procedure and to investigate how the FD time accuracy is maintained after coefficient quantization, we have designed and simulated all the five realizations considered in Sect. V (11th order TF realizing \( D=11.2 \)). The phase delay responses of the quantized TFs are given in Fig. 13 (without these of 2GM2+3IS+SC, almost fully coinciding with 2GM2+3IS+MH1, as it might be anticipated from Fig. 7). The higher overall sensitivity of the 2KW2A+2KW2B+IS+SV-structure (\( WS_{\text{max}}=669 \) in Fig. 7) is the reason for its poor performance in a limited wordlength environment – the phase delay error for low frequencies is considerable even after a mild quantization down to 4 bits (11.235 instead of 11.2 in Fig. 13a) and this response is almost totally destroyed for 2 bits wordlength. For the best structure (2MH2A+3IS+MH1) this error is almost negligible – 11.195 instead of 11.2 (Fig. 13d) and is quite acceptable even for wordlength of only 2 bit. The other structures from Sect. V are behaving as it could be predicted from Fig. 7. The main conclusion from these experiments is that our approach is working very successfully and is ensuring a considerable improvement of the accuracy in a limited wordlength environment.

In order to observe and compare the tuning accuracy of the three methods and variable structures from Sect. VI (gathering structure, Cho-Parhi-structure and IS-structure), we have designed 3 second order allpass FD filters with third order TF-coefficients approximation \( (J = 3, P = 3, T = 3) \) and a given fractional delay parameter value \( d = 0.3 \). The results after the coefficient quantization are given in Fig. 14. Because of the lower sensitivity of the IS structure the tuning accuracy is
higher than that of the gathering structure and Cho-Parhi structure even when the TF coefficients are quantized to 2 significant bits (in CSD code). The deviations from the desired phase delay (0.3 samples) of variable IS FD filter near DC for 4, 3 and 2 bits are correspondingly smaller than -0.002 and -0.009 and -0.041 and of the Cho-Parhi-structure – -0.0018, -0.0086 and -0.041.

As the RDI-method is not connected to a specific structure, we have compared its accuracy to our method by simulating the tuning of the FD from 4.1 to 4.5 of the TFs with \( N = 4 \). For our method a direct-form structure was used and the coefficients have been approximated by third-order Taylor polynomials. It is seen from Fig. 15 that the phase-delay of the RDI TF is having a higher error compared to that of our method and is losing its maximally-flat behavior for all intermediate values of \( D \) (note that for \( D=4.5 \) there is no tuning in the case of RDI-method and thus no error will appear). It was found, additionally, that there is no direct connection between the desired value of the phase-delay \( D \) and the value of the tuning factor \( \rho \) (23) and this uncertainty in tuning cannot be avoided.

**IX. CONCLUSION**

In this work a high accuracy of implementation (including multiplierless) and tuning of allpass-based fractional delay filters was achieved through sensitivity minimization. The method is based on a phase-sensitivity minimization of each individual first- and second-order allpass section in the filter cascade realization. It was shown that the poles of the FD TFs are taking positions not typical for the conventional filters. Then, after studying the possible combinations of real and complex-conjugated poles for different values of the FD parameter \( D \) and of the TF order \( N \), it was proposed to divide the unit-circle to 11 zones and it was shown that FD TF poles of most practical cases are located only in four of them and
very often – in only one (zone 2). The behavior of the most popular allpass sections when having poles in these zones was investigated and it was shown that the proper selection of the sections is very important when trying to minimize the overall sensitivity. A new second-order allpass section, providing low sensitivity for zone 2 (and thus very suitable for high accuracy FD realizations) was developed by the authors. This section was turned also to tunable and high tuning accuracy was achieved. A new approach to obtain tunable allpass FD filters was developed and it was compared with the other known methods. It was shown also that the low sensitivity so obtained permits a very short coefficient wordlength, i.e. efficient multiplierless implementations, higher processing speed and lower power consumption.

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REFERENCES


